

# MATHEMATICS

## Paper – II

Time Allowed : **Three Hours**

Maximum Marks : **200**

### Question Paper Specific Instructions

*Please read each of the following instructions carefully before attempting questions :*

*There are **EIGHT** questions in all, out of which **FIVE** are to be attempted.*

*Questions no. 1 and 5 are **compulsory**. Out of the remaining **SIX** questions, **THREE** are to be attempted selecting at least **ONE** question from each of the two Sections A and B.*

*Attempts of questions shall be counted in sequential order. Unless struck off, attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question-cum-Answer Booklet must be clearly struck off.*

*All questions carry equal marks. The number of marks carried by a question/part is indicated against it.*

*Unless otherwise mentioned, symbols and notations have their usual standard meanings.*

*Assume suitable data, if necessary, and indicate the same clearly.*

*Answers must be written in **ENGLISH** only.*

## SECTION A

- Q1.** (a) Prove that a subgroup of a cyclic group is cyclic. Let  $G$  be a cyclic group with generator  $a$ . If the order of  $G$  is infinite, then prove that  $G$  is isomorphic to  $(\mathbb{Z}, +)$ . 8
- (b) Find the relative extrema of the function  
 $f(x, y) = 4y^3 + x^2 - 12y^2 - 36y + 2$ . 8
- (c) Prove that in the interval  $0 < x < 1$ , the function  $f(x) = x^2$  is uniformly continuous while  $f(x) = \frac{1}{x}$  is not uniformly continuous. 8
- (d) Prove that  $x_1 = 2, x_2 = 1, x_3 = 0$  is a feasible solution to the following set of equations :
- $$2x_1 - x_2 + 3x_3 = 3$$
- $$-6x_1 + 3x_2 + 7x_3 = -9$$
- Is the solution basic? Justify your answer. If the solution is not basic, reduce it to a basic feasible one. 8
- (e) Find a bilinear transformation which maps the points  $z = 0, -i, -1$  into  $w = i, 1, 0$  respectively. 8
- Q2.** (a) (i) Prove that every group is isomorphic to a group of permutations. 5
- (ii) Let  $A = \{1, 2, 3\}$  and let  $S_3$  denote the symmetric group on 3 elements. Then is  $S_3$  an abelian or non-abelian group? 5
- (b) (i) Find the volume of the region above the  $xy$ -plane bounded by the paraboloid  $z = x^2 + y^2$  and the cylinder  $x^2 + y^2 = a^2$ . 7
- (ii) Prove that  $\lim_{M \rightarrow \infty} \int_0^M \frac{dx}{x^4 + 4} = \frac{\pi}{8}$ . 8

(c) (i) Let  $f(z) = \ln(1 + z)$ . Expand  $f(z)$  in a Taylor series about  $z = 0$ . Determine the region of convergence of the series. 8

(ii) Find Laurent series about the indicated singularity for the function,

$$\frac{e^z}{(z-1)^2}; z = 1. \quad 7$$

**Q3.** (a) (i) Prove that if  $u_n(x)$ ,  $n = 1, 2, 3, \dots$  are continuous in  $[a, b]$  and if  $\sum u_n(x)$  converges uniformly to the sum  $S(x)$  in  $[a, b]$ , then  $S(x)$  is continuous in  $[a, b]$ . 5

(ii) Prove that an absolutely convergent series is convergent. Show that  $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$  is conditionally convergent. 5

(b) (i) If  $N$  is a normal subgroup of a group  $G$  and if  $H$  is any subgroup of  $G$ , then prove that

$$H \vee N = HN = NH$$

where  $H \vee N$  denotes the join of  $H$  and  $N$ . 8

(ii) State the Second Isomorphism Theorem of groups and apply it to the case  $G = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ ,  $H = \mathbb{Z} \times \mathbb{Z} \times \{0\}$  and  $N = \{0\} \times \mathbb{Z} \times \mathbb{Z}$ . 7

(c) Consider the LPP :

Minimize

$$z = 10x_1 + 2x_2$$

subject to

$$x_1 + 2x_2 + 2x_3 \geq 1$$

$$x_1 - 2x_3 \geq -1$$

$$x_1 - x_2 + 3x_3 \geq 3,$$

$$x_i \geq 0, \text{ for } i = 1, 2, 3.$$

Solve the dual of the above LPP and find the minimum value of  $z$ . 15

**Q4.** (a) (i) State and prove Cauchy's integral formula. Thus evaluate

$$\oint_C \frac{\cos z}{z - \pi} dz,$$

where C is the circle  $|z - 1| = 3$ .

8

(ii) State the Residue Theorem and apply it to evaluate

$$\oint_C \frac{e^z dz}{(z - 1)(z + 3)^2}$$

where C is given by  $|z| = \frac{3}{2}$ .

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(b) Prove that the integral domain  $\mathbb{Z}$  is a Unique Factorization Domain and a Euclidean Domain.

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(c) Five workers perform five jobs and the operating cost is given below, but there is a restriction that the worker C cannot perform the third job and B cannot perform the fifth job. Find the optimal assignment and the optimal assignment cost.

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	I	II	III	IV	V
A	24	29	18	32	19
B	17	26	34	22	-
C	27	16	-	17	25
D	22	18	28	30	24
E	28	16	31	24	27

## SECTION B

- Q5.** (a) Consider a particle of mass  $m$  moving in a plane under attractive force  $\frac{k}{r^2}$  directed towards the origin, where  $k > 0$ . Using the polar coordinates  $(r, \theta)$  write the corresponding Lagrangian and obtain the equations of motion. Also show that the angular momentum is conserved. 8

- (b) A function  $f$ , defined on  $[0, 1]$ , is such that  $f(0) = 0$ ,  $f\left(\frac{1}{2}\right) = -1$ ,  $f(1) = 0$ . Find the quadratic polynomial  $p(x)$  which agrees with  $f(x)$  for  $x = 0, \frac{1}{2}, 1$ .

If  $\left| \frac{d^3 f}{dx^3} \right| \leq 1$  for  $0 \leq x \leq 1$ , show that  $|f(x) - p(x)| \leq \frac{1}{12}$  for  $0 \leq x \leq 1$ . 8

- (c) Draw the logic circuit which realises the Boolean function  $L = (A + B) \cdot (A + C) + C(A + B \cdot C)$  and simplify it. Draw the simplified circuit also. 8

- (d) In a 2-dimensional flow there are sources at  $(a, 0)$ ,  $(-a, 0)$  and sinks at  $(0, a)$ ,  $(0, -a)$ , all are of equal strength. Determine the stream function and show that the circle through these four points is a streamline. 8

- (e) Solve

$$u_{xx} + \frac{10}{3} u_{xy} + u_{yy} = -\sin(x + y) \quad 8$$

- Q6.** (a) Find the solution of

$$u_x - uu_y + u = 0$$

for the initial values  $x_0(s) = 0$ ,  $y_0(s) = s$ ,  $u_0(s) = -2s$ .

Does the solution break down for any finite  $x$ ? Is the solution unique? 15

- (b) Find a root of the equation  $\sin x + \cos x = 1$ , lying in  $(0, 2)$ , by Regula-Falsi method, correct up to four significant digits. 10

- (c) For a dynamical system having two degrees of freedom, the Lagrangian is given by  $L = \frac{m}{2} (a^2 \dot{q}_1^2 + \dot{q}_2^2) - \frac{k}{2} (a^2 + q_2^2)$ , where  $q_1$  and  $q_2$  are generalized coordinates.

Find the corresponding Hamiltonian and derive the Hamiltonian equations of motion.

Show further that the generalized momentum corresponding to  $q_1$  is constant.

Show that the system exhibits a simple harmonic motion with respect to the generalized coordinate  $q_2$ .

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- Q7.** (a) Solve :

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$$u_{tt} - u_{xx} = 0, \quad 0 < x < z, \quad t > 0$$

$$u(0, t) = u(2, t) = 0,$$

$$u(x, 0) = \sin^3 \frac{\pi x}{2},$$

$$u_t(x, 0) = 0.$$

- (b) Write down the flow-chart of Runge-Kutta method of 4<sup>th</sup> order to find  $y(0.8)$  for  $\frac{dy}{dx} = xy$ ,  $y(0) = 2$ , taking  $h = 0.2$ .

Also solve the above IVP to find  $y(0.4)$  by Runge-Kutta method (4<sup>th</sup> order).

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- (c) Consider 2-dimensional Navier-Stokes equations of a steady fluid flow.

Show that there exists a stream function  $\Psi(x, y)$  for such a flow.

Find the equation satisfied by  $\Psi(x, y)$ .

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Q8. (a) Show that

$$f(x, y, z, p, q) = x^2p^2 + y^2q^2 - 4 = 0$$

$$\text{and } g(x, y, z, p, q) = qy - a = 0,$$

where  $a$  is a constant, are compatible and hence solve  $f(x, y, z, p, q) = 0$ .

Is it complete integral ?

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(b) State the sufficient condition for convergence of the Gauss-Seidel iteration method and solve the following system of equations by using this method :

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$$6.7x_1 + 1.1x_2 + 2.2x_3 = 20.5$$

$$2.1x_1 - 1.5x_2 + 8.4x_3 = 28.8$$

$$3.1x_1 + 9.4x_2 - 1.5x_3 = 22.9$$

(correct up to 3-significant digits)

(c) There is a doublet at  $(c, 0)$  in a 2-dimensional flow. A cylinder of radius  $a$  ( $a < c$ ) with  $z$ -axis as axis of the cylinder was introduced into the flow. Find the complex potential and image system for the flow.

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