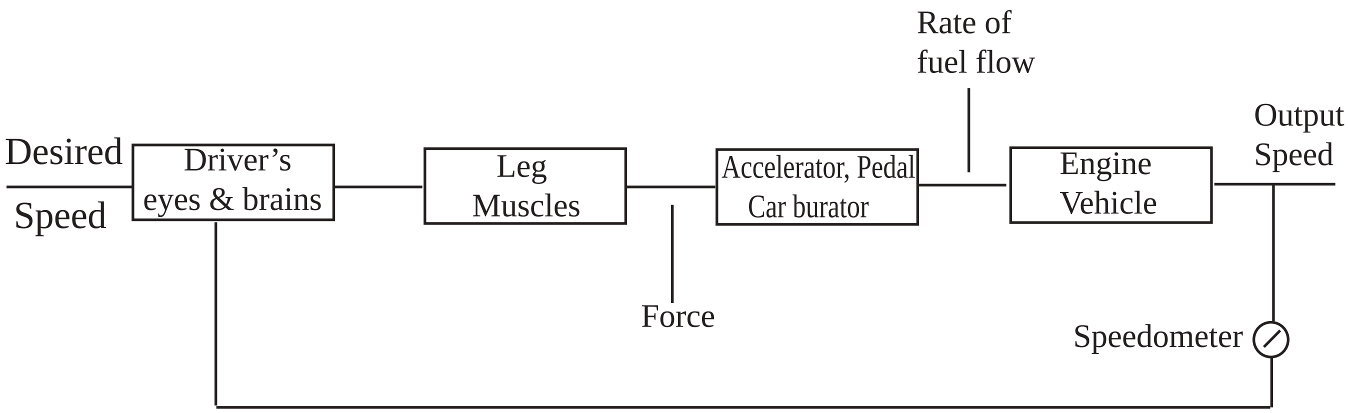
CONTROL SYSTEM

The control system is that means by which any quantity of interest in a machine, mechanism or other equation or altered in accordance which a desired manner.



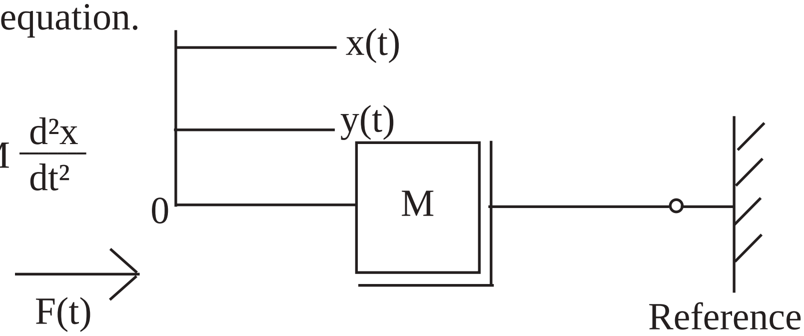
Scematic Diagram of Manually Controlled Closed Loop System

Mathematical Modeling

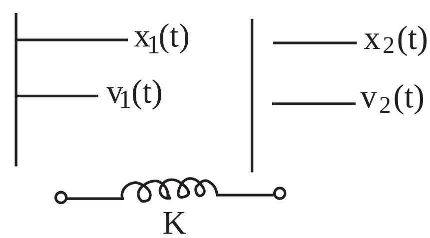
* The Differential Equaion of the system is formed by replacing each element by corresponding differential equation.

For Mechanical systems

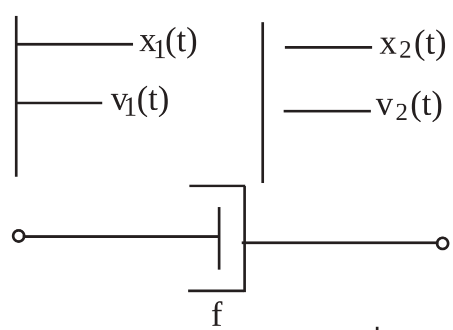
(1) $F = M\,{}^{dv}\!/_{dt} = M\dfrac{d^2x}{dt^2}$



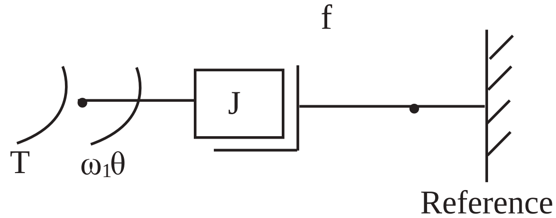
(2) $F = K(x_1 - x_2) = \int_{-\infty}^{t} K(v_1 - v_2)\,dt$



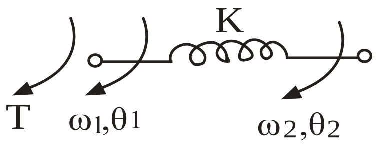
(3) $F = F(v_1 - v_2) = F\left(\dfrac{dx_1}{dt} - \dfrac{dx_2}{dt}\right)$



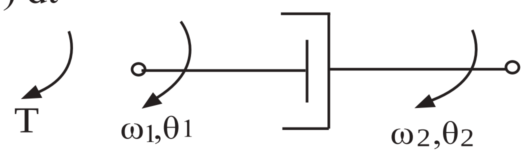
(4) $T = \dfrac{Jd\omega}{dt} = \dfrac{Jd^2\theta}{dt^2}$



(5) $T = \frac{Jd\omega}{dt} = \frac{Jd^2\theta}{dt^2}$



(6) $T = K(\theta_1 - \theta_2) = K\int_{-\infty}^{t}(\omega_1 - \omega_2)\,dt$



Analogy between Electrical & Mechanical systems

* Force (Torque) - Voltage Analogy

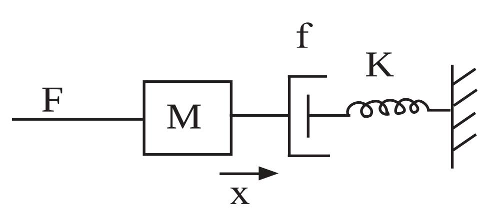
Translation system	Rotational system	Electrical system
Force F	Torque T	Voltage e
Mass M	Moment of Inertia J	Inductance L
Visuals Friction coefficient f	Viscous Friction coefficient f	Resistance R
Spring stiffness K	Tensional spring stiffness K	Reciprocal of capacitance $^1/_C$
Displacement x	Angular Displacement θ	Charge q
Velocity v	Angular velocity ω	Current i

* Force (Torque) - Current Analogy

Translation system	Rotational system	Electrical system
Force F	Torque T	Current i
Mass M	Moment of Inertia J	Capacitance C
Visuals Friction coefficient f	Viscous Friction coefficient f	Reciprocal of Resistant 1/R
Spring stiffness K	Tensional spring stiffness K	Reciprocal of Inductance $^1/_L$
Displacement x	Angular Displacement θ	Magnetic flux linkage λ
Velocity v	Angular velocity ω	Voltage e

Trasfer function

The differential equation for this system is



$$F = M\frac{d^2x}{dt^2} + f\frac{dx}{dt} + kx$$

Take Laplace Transform both sides

$F(s) = Ms^2X(s) + fsX(s) + kX(s)$ [Assuming zero initial conditions]

$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + fs + k} = \text{Transfer function of the system}$$

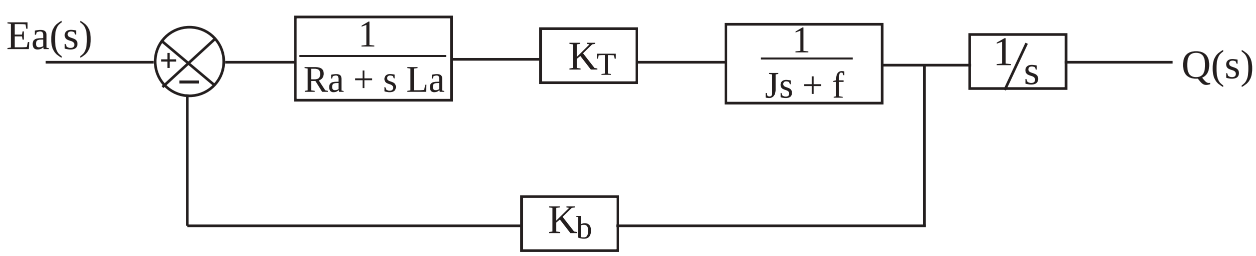
Transfer function is ratio of Laplace Transform of output variable to Laplace Tansform of input variable.

* The steady state-response of a control system to a sinusoidal input is obtained by replacing 's' with 'jw' in the transfer function of the system.

$$\frac{X(jw)}{F(jw)} = \frac{1}{M(jw)^2 + f(jw) + k} = \frac{1}{-w^2M + jwf + K}$$

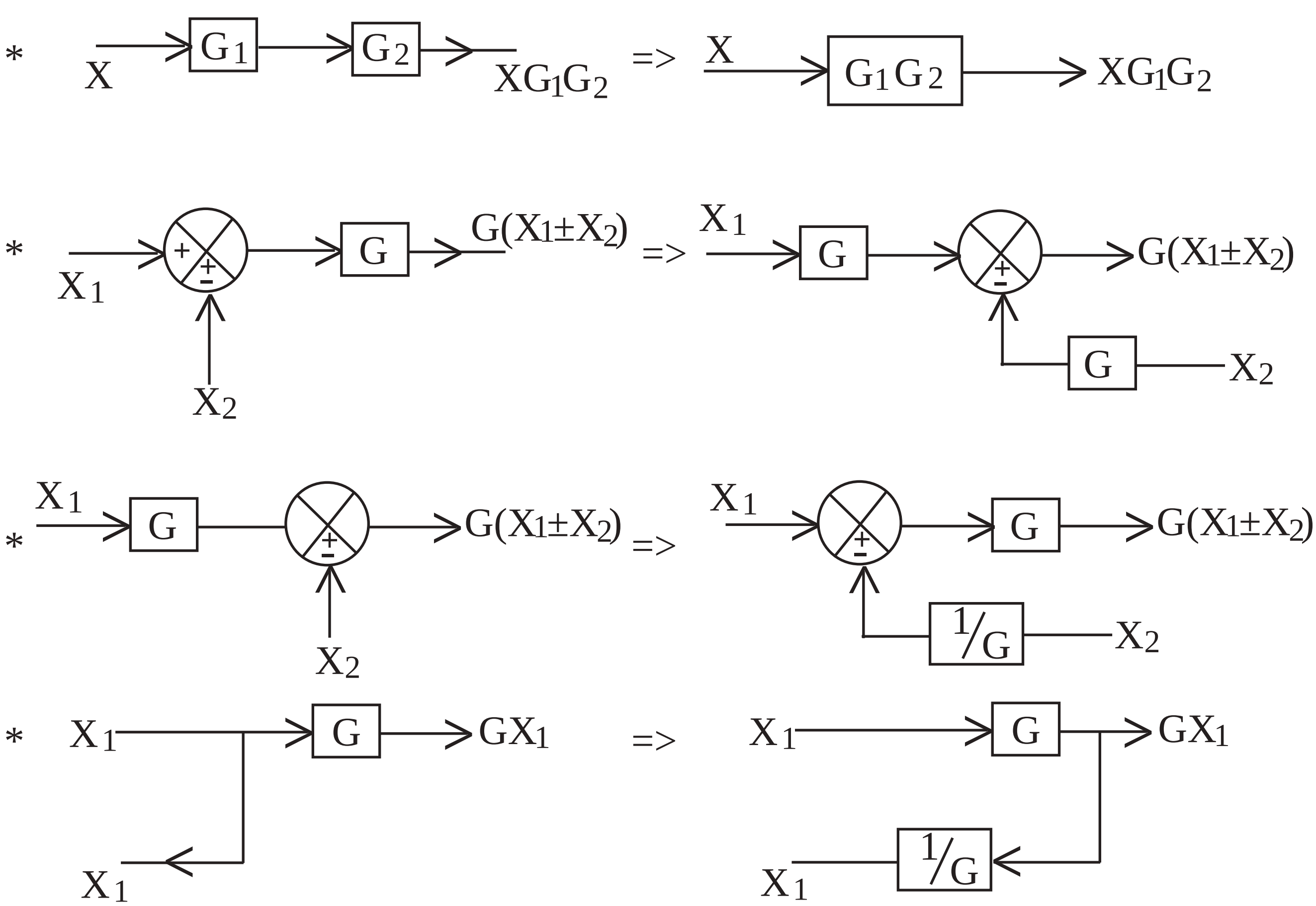
Block - Diagram Algebra

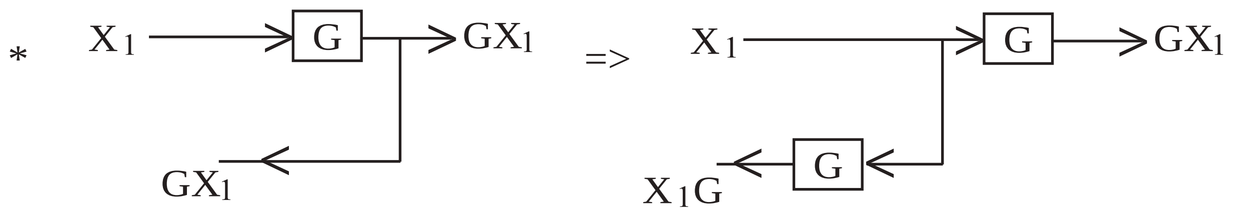
The system can also be represented graphical with the help of block diagram.

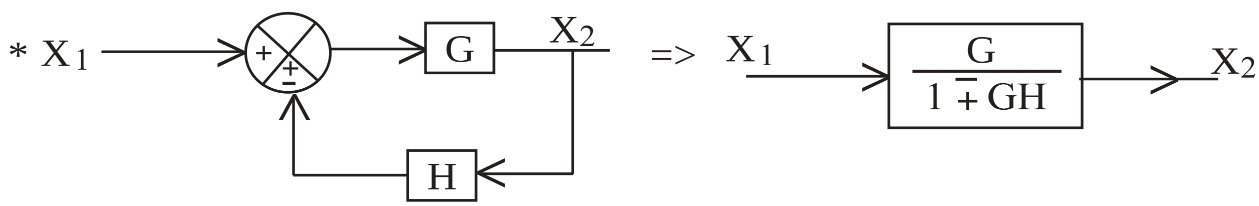


Block Diagram For Armature Control Of DC Motor

Various blocks can be replaced by a signal block to simplify the block diagram.







Signal Flow Graphs

- * **Node:** It represents a system variable which is equal to sum of all incoming signals at the node. Outgoing signals do not affect value of node.
- * **Branch:** A signal travels along a branch from one node to another in the direction indicated by the branch arrow & in the process gets multiplied by gain or transmittance of branch.
- * **Forward Path:** Path from input node to output node.
- * **Non-Touching loop:** Loops that do not have any common node.

Mason's Gain Formula

Ratio of output to nput variable of a signal flow graph is called net gain.

$$T = \frac{1}{\Delta} \sum_K P_k \Delta_k \qquad P_k = \text{path gain of k th forward path}$$

Δ = determinant of graph = 1 - (sum of gain of individual loops)
+ (sum of gain product of 2 non touching loops)
- (sum of gain product of 3 non touching loops)+...

$$= 1 - \sum_K P_{m1} + \sum P_{m2} - \sum P_{m3} + \ldots\ldots\ldots\ldots$$

P_{mr} = gain product of all 'r' non touching loops.
Δ_K = the value of Δ for the part of graph not touching k th forward path.
T = overall gain
Example:

Forward Paths:

$P_1 = a_{12}\, a_{23}\, a_{34}\, a_{45}$

$P_2 = a_{12}\, a_{23}\, a_{35}$

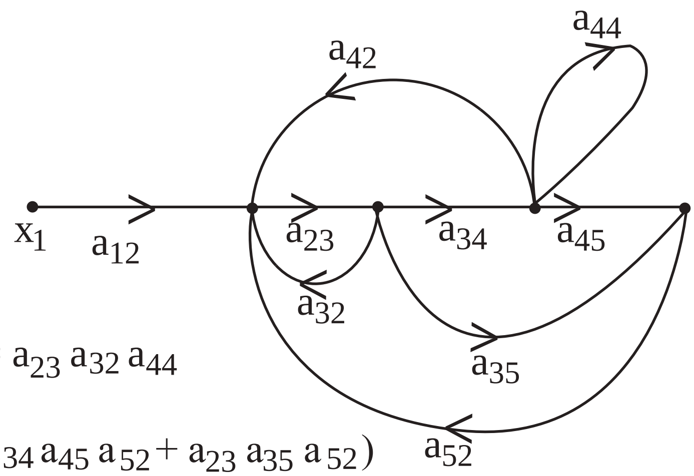
Loops: $P_{11} = a_{23}\,a_{32}$

$P_{21} = a_{23}\,a_{34}\,a_{42}$

$P_{31} = a_{44}$

$P_{41} = a_{23}\,a_{34}\,a_{45}\,a_{52}$

$P_{51} = a_{23}\,a_{35}\,a_{52}$



2-Non - Touching loops

$P_{12} = a_{23}\,a_{32}\,a_{44}$; $P_{12} = a_{23}\,a_{32}\,a_{44}$

$$\Delta = 1 - (a_{23}a_{32} + a_{23}a_{42} + a_{44} + a_{23}a_{34}a_{45}a_{52} + a_{23}a_{35}a_{52}) + (a_{23}a_{32}a_{44} + a_{23}a_{35}a_{52}a_{44})$$

First forward path is in touch with all loops $\Delta_1 = 1$

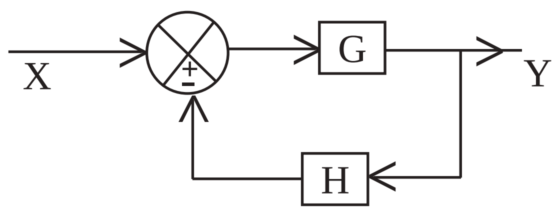
Second forward path does not touch one loop $\Delta_1 = (1 - a_{44})$

$$T = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} = \frac{a_{12}a_{23}a_{34}a_{45} + a_{12}a_{23}a_{35}\,(1 - a_{44})}{\Delta}$$

Effect of Feed back

System before feed back

System after feed back



Effect on Gain

Positive feedback

$$\text{Gain} = \frac{G}{1 - GH} > G \ \text{(gain increases)}$$

Negative feedback

$$\text{Gain} = \frac{G}{1 - GH} < G \ \text{(gain decreases)}$$

Effect on Stability

Feedback can improve stability or be harmful to stability if not applied properly.

Eg. $\text{Gain} = \dfrac{G}{1 + GH}$ & $GH = -1$, output is infinite for all inputs.

Effect on Sensitivity

Sensitivity is the ratio of relative change in output to relative change in input.

$$S_G^T = \frac{\partial T/T}{\partial G/G} = \frac{\partial \mathrm{Ln}T}{\partial \mathrm{Ln}G}$$

For open loop system $T = G$

$$S_G^T = \frac{G}{T}\frac{\partial T}{\partial G} = \frac{G}{G} \times 1 = 1$$

For closed loop system $T = \dfrac{G}{1+GH}$

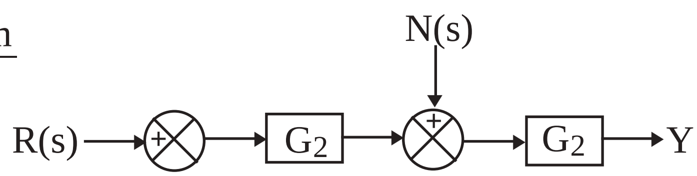
$$S_G^T = \frac{G}{T}\frac{\partial T}{\partial G} = \frac{G(1+GH)}{G} \times \frac{1}{(1+GH)^2} = \frac{1}{1+GH} < 1 \text{ (Sensitivity decreases)}$$

Effect on Noise :

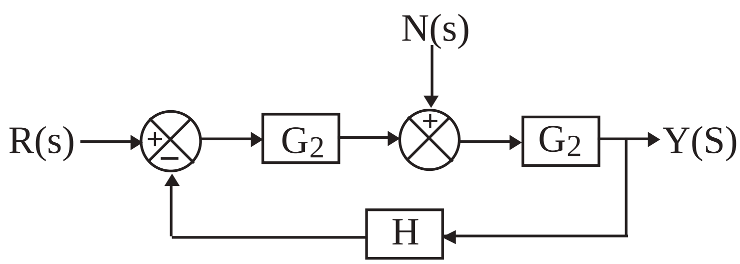
Feedback can reduce the effect noise and disturbance on system's performance.

Open - Loop System

$$\frac{Y(S)}{N(S)} = G_2$$



Closed - Loop System



$$\frac{Y(S)}{N(S)} = \frac{G_2}{1+G_1G_2H} < G_2 \quad \text{(Effect of Noise Decreases)}$$

* Positive feedback is mostly employed in oscillator whereas negative feedback is used uin amplifiers.

Time Response Analysis

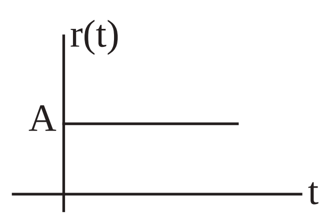
* Standard Test signals

→ Step signal

$r(t) = 1; \; t > 0$

$u(t) = 1; \; t > 0 = 0; \; t < 0$

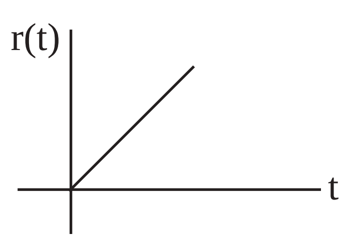
$R(s) = A/s$



→ Ramp Signal

$r(t) = At, \; t > 0 = 0, \; t < 0$

$R(s) = A/s^2$

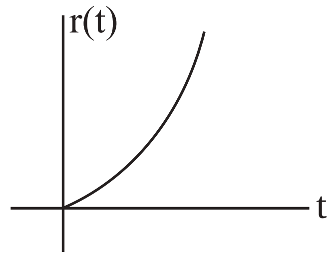


→ **Parabolic signal**

$$r(t) = \frac{At^2}{2} \quad ; \quad t > 0$$

$$= 0 \; ; \; t < 0$$

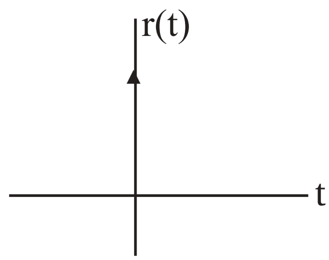
$$R(s) = \frac{A}{s^3}$$



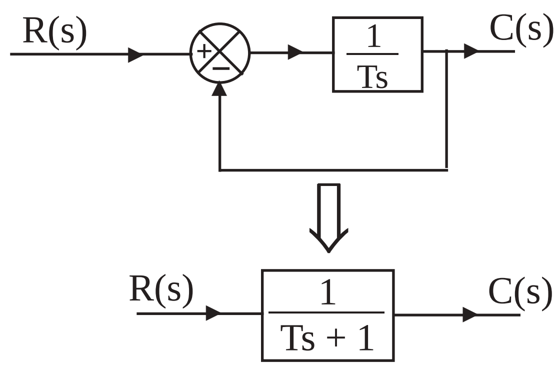
→ **Impulse**

$$\delta(t) = 0 \; ; \; t \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$



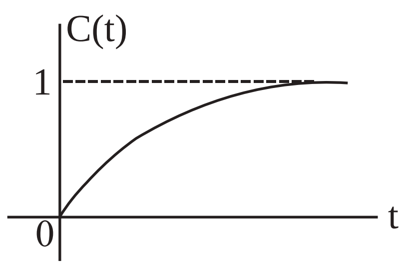
* **Time Response of first - order systems**



→ Unit step input $\quad R(s) = \frac{1}{S}$

$$C(s) = \frac{1}{S(Ts+1)} = \frac{1}{S} - \frac{T}{Ts+1}$$

$$C(t) = 1 - e^{-t/T}$$

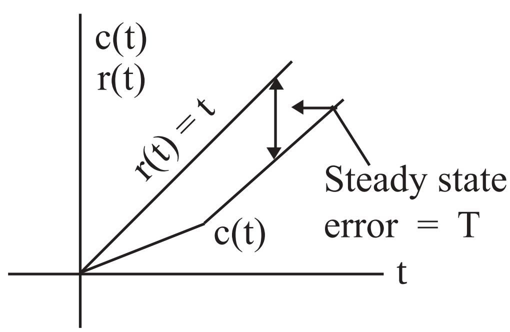


→ Unit Ramp input

$$R(s) = \frac{1}{S^2}$$

$$C(s) = \frac{1}{S^2(Ts+1)}$$

$$C(t) = t - T\left(1 - e^{-t/T}\right)$$



Type of System

Steady state error of system (e_{ss}) depends on number of poles of G(s) at s = 0. This number is known as types of system.

Error Constants

For unity feedback control systems.

K_P (position error constant) $= \lim_{s \to 0} G(s)$

K_V (Velocity error constant) $= \lim \ s\,G(s)$

K_a (Acceleration error constant) $= \lim\limits_{s \to 0} s^2 G(s)$

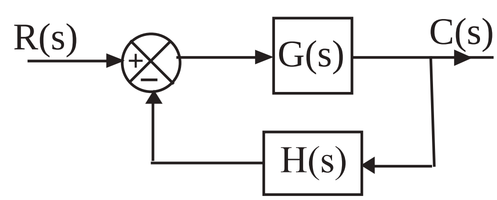
Steady state error for unity feedback systems.

Type of system j	Error constants K_P K_V K_a	Step input $\frac{R}{1 + K_P}$	Ramp input $\frac{R}{K_V}$	Parabolic input $\frac{R}{K_a}$
0	K 0 0	$\frac{R}{1+K}$	∞	∞
1	∞ K 0	0	$\frac{R}{K}$	∞
2	∞ ∞ K	0	0	$\frac{R}{K}$
3	∞ ∞ ∞	0	0	0

* For non-unity feedback systems, the difference between input signal R(s) and feedback signal B(s) actuating error signal $E_a(s)$.

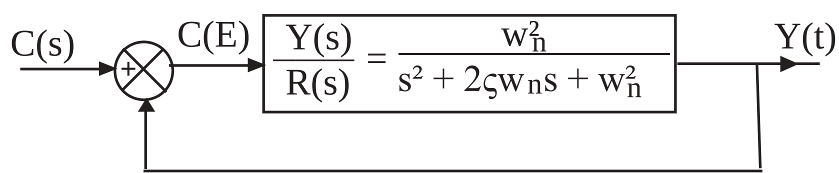
$$E_a(s) = \frac{1}{1 + G(s)H(s)} R(s)$$

$$e_{a\,ss} = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)H(s)}$$



Transient Response of second order system

$$G(s) = \frac{w_n^2}{S(S + 2\varsigma w_n)}$$



$$\frac{Y(s)}{R(s)} = \frac{w_n^2}{s^2 + 2\varsigma w_n s + w_n^2}$$

Characteristic Equation : $\Delta(s) = s^2 + 2\zeta w_n s + w_n^2)$

For unit step input

$$Y(s) = \frac{w_n^2}{s(s^2 + 2\varsigma w_n s + w_n^2)}$$

If $\varsigma < 1$ (under damp) $\qquad y(t) = 1 - \frac{e^{-\zeta w_n t}}{\sqrt{1 - \zeta^2}} \sin (w_d t + \psi)$

$w_d = w_n \sqrt{1 - \varsigma^2}$; $\psi = \tan^{-1} \frac{\sqrt{1 - \zeta^2}}{\zeta}$

If $\varsigma = 1$ (critical damp) $\qquad y(t) = 1 - (1 + w_n t)\, e^{-w_n t}$

If $\varsigma > 1$ (over damp)

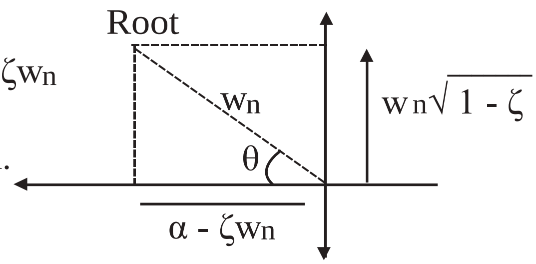
$$y(t) = 1 - \left[\cos h\left(w_n\sqrt{\zeta^2 - 1}\right)t - \frac{\zeta^2}{\sqrt{\zeta^2 - 1}}\sin h\left(w_n\sqrt{\zeta^2 - 1}\right)t\right]e^{-w_n t}$$

Roots of characteristic equation are

$s_1, s_2 = \zeta w_n \pm j w_n\sqrt{1 - \zeta^2}$ $\alpha = \zeta w_n$

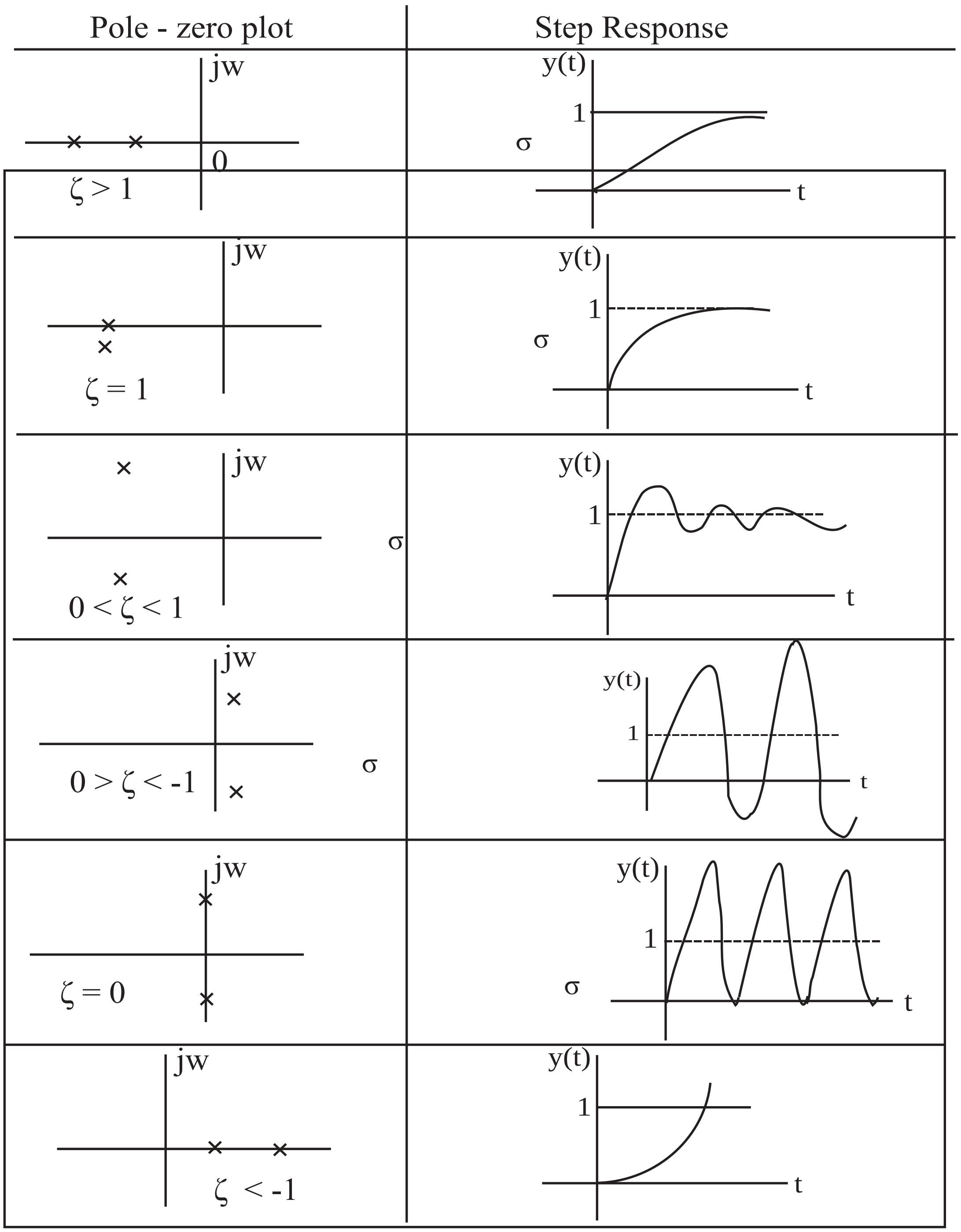
'α' is damping constant which governs decay of response for under damped system.

$\zeta = \cos\theta.$



$\zeta = 0$, imaginary axis

If correspond to "undamped system" or sustained oscillations.



Important Characteristic of Step Response

* Maximum overshoot : $100e^{-\pi\zeta/\sqrt{1-\zeta^2}}$ %

* Rest time: $\dfrac{\pi - \tan^{-1}\left(\dfrac{\sqrt{1-\zeta^2}}{\zeta}\right)}{w_n\sqrt{1-\zeta^2}}$ * Peak time : $\dfrac{\pi}{w_n\sqrt{1-\zeta^2}}$

* Setting Time: $t_s = \dfrac{4}{\zeta w_n}$ (for 2% margin)

$t_s = \dfrac{3}{\zeta w_n}$ (for 5% margin)

Effect of Adding poles and zeroes to Transfer Function

1. If a pole is added to forward transfer function of a closed loop system, it increases maximum overshoot of the system.
2. If a pole is added to closed loop transfer function it has effect opposite to that of case - 1.
3. If a zero is added to forward path transfer function of a closed loop system, it decreases rise time and increases maximum overshoot.
4. If a zero is added to closed loop system, rise time decreases but maximum overshoot decreases than increases as zero added moves towards origin.

Stability of Control System

A linear, time-invariant system is stable if following notions of stability are satisfied:

* When system is excited by bounded input, the output is bounded.
* In absence of inputs, output tends towards zero irrespective of initial conditions.
* For system of first and second order, the positive ness of coefficients of characteristic equation is sufficient condition for stability.
* For higher order systems, it is necessary but not sufficient condition for stability.

Routh Stability Criterion

* If is necessary & sufficient condition that each term of first column of Routh Array of its characteristic equation is positive if $a_0 > 0$.
* Number of sign changes in first column = Number of roots in Right Half Plane.

* Example: $a_0 s^n + a_1 s^{n-1} + \dots\dots\dots + a_n = 0$

s^n	a_0	a_2	a_4	....
s^{n-1}	a_1	a_3	a_5	....
s^{n-2}	$\frac{a_1 a_2 - a_0 a_3}{a_1}$		$\frac{a_1 a_4 - a_0 a_5}{a_1}$	
s^{n-3}	..			
..	..			
..	..			
s^0	a_n			

Special Cases

* When first term in any row of the Routh Array is zero while the row has at least one non-zero term.

Solution: Substitute a small positive number '$\in$' for the zero& proceed to evaluate rest of Routh Array

eg. $s^5 + s^4 + 2s^3 + 2s^2 + 3s + 5 = 0$

s^5	1	2	3
s^4	1	2	5
s^3	$\in$	-2	
s^2	$\frac{2\in + 2}{\in}$	5	
s^1	$\frac{-4\in - 4 - 5\in^2}{2\in + 2}$	$\rightarrow$ -2	
s^0	5		

$\frac{2\in + 2}{\in} > 0$, and hence there are 2 sign change and thus 2 roots in right half plane.

* When all the elements in any one row Routh Array are zero.

Solution: The polynomial whose coefficients are the elements of row just above row of zeroes in Routh Array is called Auxiliary polynomial.

* The order of auxiliary polynomial is always even.

* Row of zeroes should be replaced row of coefficients of polynomial generated by taking first derivative of auxiliary polynomial.

Example: $s^6 + 2s^5 + 8s^4 + 12s^3 + 20s^2 + 16s + 16 = 0$

s^6	1	8	20	16	s^6	1	8	20	16
s^5	2	12	16		s^5	1	6	8	
s^5	1	6	8		s^4	1	6	8	
s^4	2	12	16		s^3	4	12		
s^4	1	6	8		s^3	1	3		
s^3	0	0			s^2	3	8		
					s^1	$1/3$			
					s^0	8			

Auxiliary polynomial : $A(s) = s^4 + 6s^2 + 8 = 0$

Types of Stability

* **Limited stable:** If non-repeated root of characteristic equation lies on jw-axis.
* **Absolutely stable:** With respect to a parameter if it is stable for all value of this parameter.
* **Conditionally stable:** With respect to a parameter if system is stable for bounded range of this parameter.

Relative Stability

If stability with respect to a line $s = \sigma_1$ is to be judged, then we replace s by $(z + \sigma_1)$ in characteristic equation and judge stability based on Routh criterion, applied on new characteristic equation.

Roots Locus Technique

Root-Loci is important to study trajectories of poles and zeroes as the poles & zeroes determine transient response & stability of the system.

Characteristic equation $1 + G(s)H(s) = 0$

Assume $G(s)H(s) = KG_1(s)H_1(s)$

$1 + KG_1(s)H_1(s) = 0$ $\quad G_1(s)H_1(s) = -1/K$

Condition of Roots Locus

$$|G_1(s)\,H_1(s)| = \frac{1}{|k|} \quad -\infty < k < \infty$$

$\angle G_1(s)H_1(s) = (2i + 1)\pi$ $\quad K \geq 0$ = odd multiple of 180°

$\angle G_1(s)H_1(s) = 2i\pi$ $\quad K \leq 0$ = even multiples of 180°

Condition for a point to lie on root Locus

* The difference between the sum of the angles of the vectors drawn from the zeroes and those from the poles of G(s) H(s) to s_1 is on odd multiple of 180° if $K > 0$.

ENTRI

* The difference between the sum of the angles of the vectors drawn from the zeroes & those from the poles of G(s)H(s) to s_1 is an even multiple of 180° including zero degrees.

Properties of Roots loci of $1 + KG_1(s)H_1(s) = 0$

1. K = 0 points : These points are poles of G(s)H(s), including those at $s = \infty$.
2. $K = \pm\infty$ point : The $K = \pm\infty$ points are the zeroes of G(s)H(s) including those at $s = \infty$.
3. Total number of Root loci is equal to order of $(1 + KG_1(s)H_1(s) = 0)$ equation.
4. The root loci are symmetrical about the axis of symmetry of the pole-zero configuration G(s) H(s).
5. For large values of s, the RL (K > 0) are asymptotes with angles given by:

$$\theta_i = \frac{2i + 1}{|n - m|} \times 180°$$

for CRL (complementary root loci) (K < 0)

$$\theta_i = \frac{2i}{|n - m|} \times 180°$$

where

$i = 0, 1, 2, ..., |n - m| - 1$, n = no. of finite poles of G(s) H(s),
m = no. of finite zeroes of G(s) H(s)

6. The intersection of asymptotes lies on the real axis in s-plane.
The point of intersection is called centroid (σ)

$$\sigma_1 = \frac{\sum \text{real parts of poles } G(s)H(s) - \sum \text{real parts of zeroes } G(s)H(s)}{n - m}$$

7. Roots locus are found in a section of the real axis only if total number of poles and zeroes to the side of section is odd if K > 0. For CRL (K < 0), the number of real poles & zeroes to right of given section is even, then that section lies on root locus.

8. The angle of departure of arrival of roots loci at a pole or zero of G(s) H(s) say s_1 is found by removing term $(s - s_1)$ from the transfer function and replacing 's' by 's1' in the remaining transfer function to calculate $\angle G(s_1)H(s_1)$
Angle of Departure (only applicable for poles) = $180° + \angle G(s_1)H(s_1)$
Angle of Arrival (only applicable for zeroes) = $180° - \angle G(s_1)H(s_1)$

9. The crossing point of root-loci on imaginary axis can be found by equating coefficient of s^1 in Routh table to zero & calculating K.
Then roots of auxiliary polynomial give intersection of root locus with imaginary axis.

10. Break-away & Break-in points
These points are determined by finding roots of $\frac{dk}{ds} = 0$

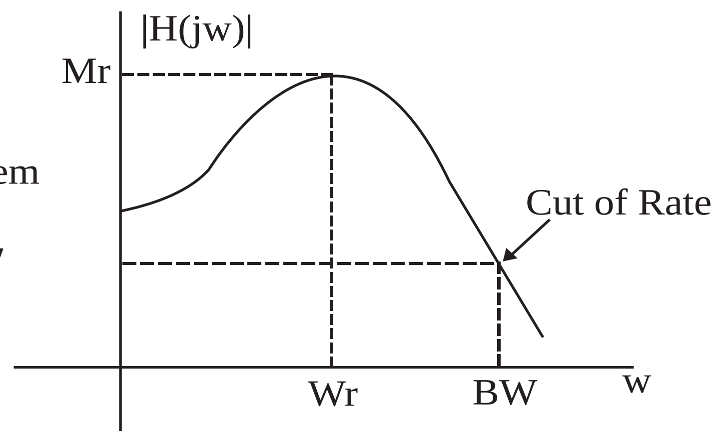
for breakaway points: $\frac{d^2k}{ds^2} < 0$ for break in points: $\frac{d^2k}{ds^2} > 0$

11. Value of k on Root locus is $|K| = \frac{1}{|G_1(s_1)H_1(s_1)|}$

Addition of poles & zeroes to G(s)H(s)

* Addition of a pole to G(s) H(s) has the effect of pushing of roots loci toward right half plane.
* Addition of left half plane zeroes to the function G(s)H(s) generally has effect of moving & bending the root loci toward the left half s-plane.

Frequency Domain Analysis



* <u>Resonant Peak, M_r</u>
It is the maximum value of |M(jw)| for second order system

$$Mr = \frac{1}{2\zeta\sqrt{1 - \zeta^2}}, \quad \zeta \leq 0.707$$

ζ = dmping coefficient

* <u>Resonant frequency, w_r</u>
The resonant frequency w_r is the frequency at which the peak M_r occurs.
$$w_r = w_n \sqrt{1 - 2\zeta^2}, \quad \text{for second order system}$$

* <u>Bandwidth, BW</u>
The bandwidth is the frequency at which |M(jw)| drops to 70.7% of, or 3 dB down from, its zero frequency value.
for second order system, $BW = w_n \left[(1 - 2\zeta^2) + \sqrt{\zeta^4 - 4\zeta^2 + 2}\right]^{\frac{1}{2}}$
Note: For $\zeta > 0.707$, $w_r = 0$ and $M_r = 1$ so no peak.

Effect of Adding poles and zeroes to forward transfer function

* The general effect of adding a zero to the forward path transfer function is to increase the bandwidth of closed loop system.

* The effect of adding a pole to the forward path transfer function is to make the closed loop less stable, which decreasing the bandwidth.

Nyquist Stability Criterion

In addition to providing the absolute stability like the Routh Hurwitz criterion, the Nyquist criterion gives information on relative stability of a stable system and the degree of instability of an unstable system.

Stability Condition

* Open - loop stability: If all poles of G(s)H(s) lie in left half plane.
* Closed loop stability: If all roots of 1 + G(s)H(s) = 0 lie in left half plane.

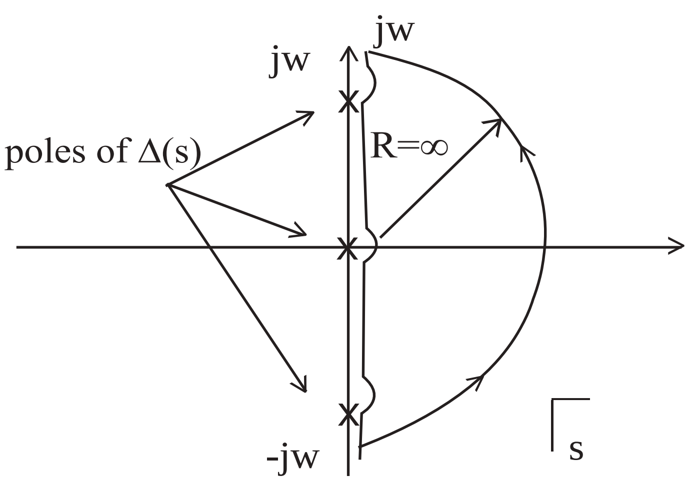
Encircled or Enclosed

A point of region in a complex plane is said to be encircled by a closed path if it is found inside the path.

A point of region is said to enclosed by a closed path if it is encircled in the counter clockwise direction, or the point or region lies to the left of path.

Nyquist Path

* If is a semi-circle that encircles entire right half plane but it should not pass through any poles or zeroes of Δ(s) = 1 + G(s)H(s) & hence we draw small semi-circles around the poles & zeroes on jw-axis.



Nyquist Criterion

1. The Nyquist path Γ_s is defined in s-plane, as shown above.
2. The L(s) plot (G(s)H(s) plot) in L(s) plane is drawn i.e., every point Γ_s plane is mapped to corresponding value of L(s) = G(s)H(s).
3. The number of encirclements N, of the (-1 + j0) point made by L(s) plot is observed.
4. The Nyquist criterion is N = Z - P

   N = number of encirclement of the (-1 + j0) point made by L(s) plot.

   Z = number of zeroes of 1 + L(s) that are inside Nyquist path (i.e., RHP)

   P = number of poles of 1 + L(s) that are inside Nyquist path (i.e., RHP); poles of 1 + L(s) are same as poles of L(s).

For closed loop stability Z must equal 0.
For open loop stability, P must equal 0.
∴ For closed loop stability $N = -P$
i.e., Nyquist plot must encircle (-1+j0) point as many times as no. of poles of L(s) in RHP but in clockwise direction.

Nyquist criterion for Minimum phase system

A minimum phase transfer function does not have poles or zeroes in the right half s-plane or on-axis excluding origin. For a closed loop system with loop transfer function L(s) that is of minimum phase type, the system is closed loop stable if the L(s) plot that corresponds to the Nyquist path does not encircle (-1 + j0) point it is unstable. i.e., $N = 0$

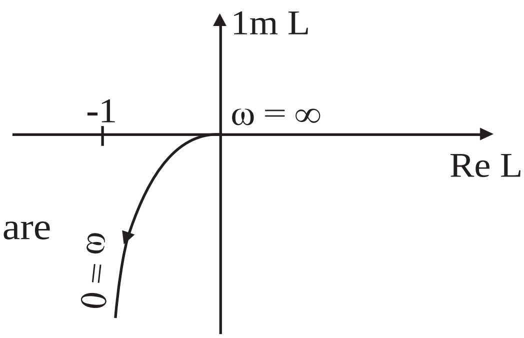
* Effect of addition of poles & zeroes to L(s) on shape of Nyquiest plot

$$\text{If } L(s) = \frac{K}{1 + T_1 s}$$

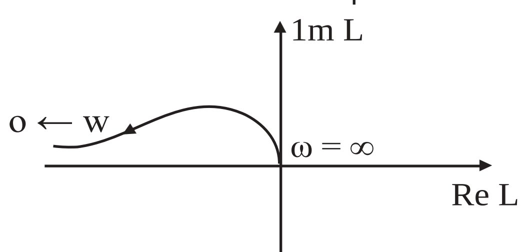
Addition of poles at s = 0

1. $L(s) = \dfrac{K}{s(1 + T_1 s)}$

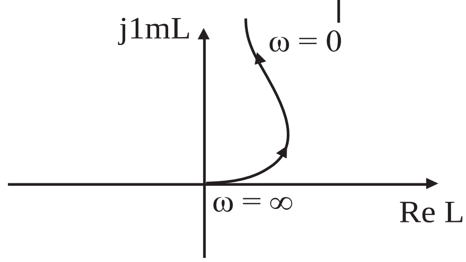
Both Head & Tail of Nyquist plot are rotated by 90° clockwise.



2. $L(s) = \dfrac{K}{s^2(1 + T_1 s)}$

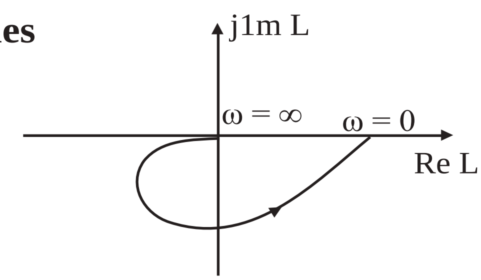


3. $L(s) = \dfrac{K}{s^3(1 + T_1 s)}$



Addition of finite non-zero poles

$$L(s) = \frac{K}{(1 + T_1 s)(1 + T_2 s)}$$



Only the head is moved clockwise by 90º but tail point remains same.

Addition of zeroes

Addition of term $(1 + T_d s)$ in numerator of L(s) increasees the phase of L(s) by 90º at $w \rightarrow \infty$ and hence improves stability.

Relative stability : Gain & Phase Margin

* **Gain Margin**

Phase crossover frequency is the frequency at which the L(jw) plot intersect the negative real axis.

or where $\angle L(jw_P) = 180^o$.

$$\text{Gain margin} = GM = 20\log_{10} \frac{1}{|L(jw_P)|}$$

If L(jw) does not intersect the negative real axis.

$$|L(jw_P)| = 0 \qquad GM = \infty dB$$

GM > 0 dB indicates stable system.

GM = 0 dB indicates marginally stable system.

GM < 0 dB indicates unstable system.

Higher the value of GM, more stable the system is.

* **Phase Margin**

It is defined as the angle in degrees through which L(jw) plot must be rotated about the origin so that gain crossover passes through (-1, j 0) point.

Gain crossover frequency is w_g s.t. $\quad |L(jw_g)| = 1$

Phase margin (PM) = $\angle L(jw_g) - 180^o$

Bode Plots

Bode plot consist of two plots

* 20 log | G (jw) | vs log w * ϕ(w) vs log w

$$\text{Assuming } G(s) = \frac{K(1 + T_1 s)(1 + T_2 s)}{s(1 + T_a s)\left(1 + 2\,{}^{\zeta s}\!/\!_{\omega_n^2} + {}^{s^2}\!/\!_{\omega_n^2}\right)} e^{-T_d s}$$

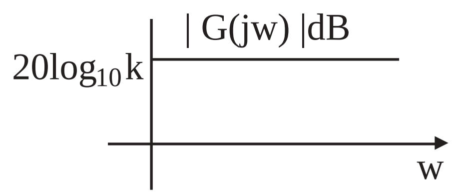
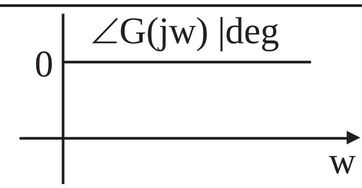
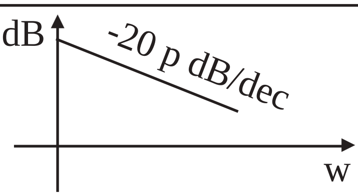
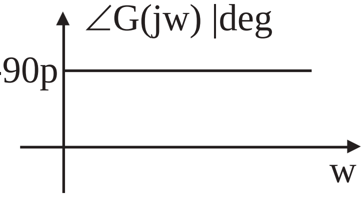
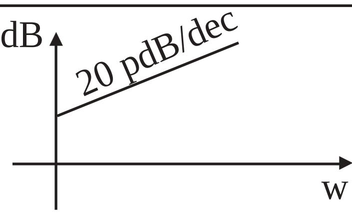
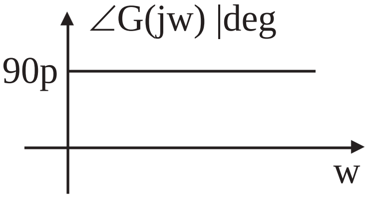
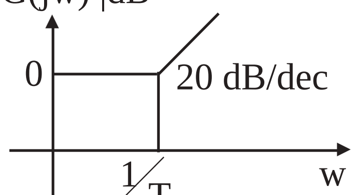
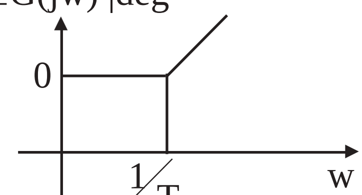
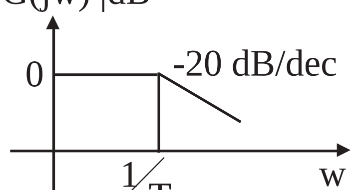
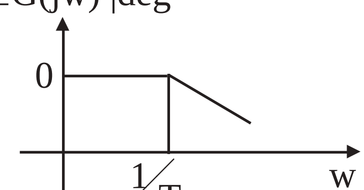
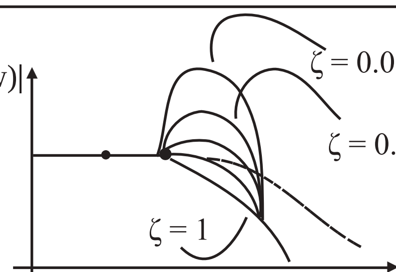
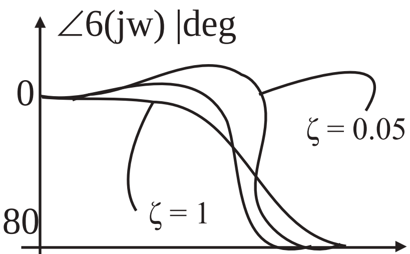
$|G(jw)|_{dB} = 20 \log |G(jw)| = 20\log_{10}|K| + 20\log_{10}|1 + jwT_1| + 20\log_{10}|1 + jwT_2| - 20\log_{10}|jw| - 20\log_{10}|1 + jwT_a|$

$$-20\log_{10}\left|1 + j2\frac{\zeta w}{w_n} - {}^{w^2}\!/\!_{w_n^2}\right|$$

$\angle G(jw) = \angle(1 + jwT_1) + \angle(1 + jwT_2) - \angle jw - \angle(1 + jwT_a) -$

$$\angle\left(1 + 2j\zeta w/w_n - {}^{w^2}\!/\!_{w_n^2}\right) - jwTd \text{ rad}$$

Magniude & Phase plot of various factor

Factor	Magnitude plot	Phase Plot
K		
$(jw)^{-P}$		
$(jw)^{P}$		
$(1 + jwT_a)$		
$(1 + jwT_a)^{-1}$		
G(jw) = $\frac{1}{\left[1 \left(\frac{w}{w_n}\right)^2\right] + j2\varsigma\left(\frac{w^2}{w_n}\right)}$		

Example : Bode plot for $G(s) = \frac{10\,(s + 10)}{s(s + 2)\,(s + 5)}$

$G(jw) = \frac{10(10 + jw)}{jw(jw + 2)\,(jw + 5)}$ If w = 0.1

$|\,G(jw)\,| = \frac{10^2}{0.1 \times 2 \times 5} = 100\,;$ For $0.1 < w < 2$

$$|G(jw)| = \frac{10^2}{w \times 2 \times 5} = 10/w \; ; \quad \text{Slope} = -20 \text{ dB/dec} \qquad \angle G(jw) = -90^o$$

For $2 < w < 5$

$$G(jw) = \frac{10 \times 10}{jw \times jw \times 5} = \frac{-20}{w^2} \; ; \; |G(jw)| \rightarrow \text{Slope} = -40 \text{ dB/dec}$$

$\angle G(jw) = 180^o$

For $5 < w < 10$

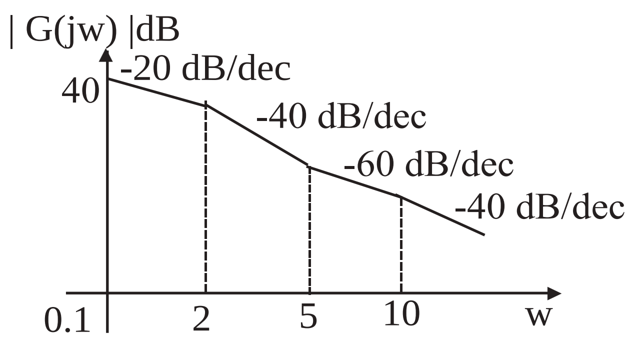
$$G(jw) = \frac{10 \times 10}{jw \times jw \times jw} = +j\frac{100}{w^3} \; ; \; |G(jw)| \rightarrow \text{Slope} = -60 \text{ dB/dec}$$

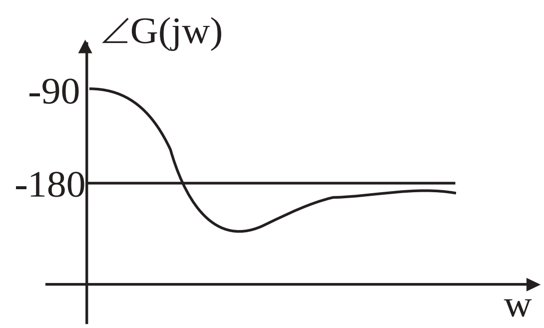
$\angle G(jw) = -270^o$

For $w > 10$

$$G(jw) = \frac{10 \times jw}{jw \times jw \times jw} = -10/w^2 \; ; \; |G(jw)| \rightarrow \text{Slope} = -40 \text{ dB/dec}$$

$\angle G(jw) = -180^o$





Designs of Control systems

* **P - Controller**

The transfer function of this controller is K_P.

The main disadvantage in P - controller is that as K_P value increases, ζ decreases & hence overshoot increases.

As overshoot increases system stability decreases.

* **I - Controller**

The transfer function of the controller is $\frac{k_i}{s}$.

It introduces a pole at origin and hence type is increased and as type increases, the SS error decrease but system stability is affected.

* **D - Controller**

It's purpose is to improve the stability.

The transfer function of this controller is $K_D S$.

It introduces a zero at origin so system type is decreased but steady state error increases.

* **PI - Controller**

It's purpose SS error without affection stability.

$$\text{Transfer function} = K_P + \frac{K_i}{s} = \frac{(SK_P + K_i)}{S}$$

It adds pole at origin, so type increases & SS error decreases.
It adds a zero in LHP, so stability is not affected.

Effects :

* Improves damping and reduces maximum overshoot.
* Increases rise time. * Decreases BW.
* Improves Gain Margin, Phase margin & Mr.
* Filter out high frequency noise.

* **PD Controller**

Its purpose is to improve stability without affecting stability.
Transfer function : $K_P + K_D S$
It adds a zero in LHP, so stability improved.

Effects :

* Improves damping and maximum overshoot.
* Reduces rise time & setting time.
* Increase BW. * Improves GM, PM, Mr.
* May attenuate high frequecy noise.

* **PID Controller**

* Its purpose is to improve stability as well as to decrease e_{SS}.

$$\text{Transfer function} = K_P + \frac{K_i}{s} + sK_D$$

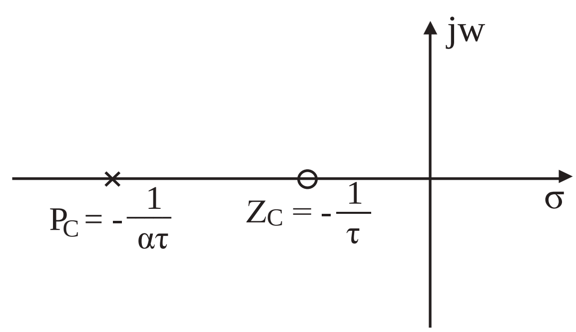
* If adds a pole at origin which increases type & hence steady stae error decreases.
* If adds 2 zeroes in LHP, one finite zero to avoid effect on stability & other zero to improve stability of system.

Compensators

* **Lead Compensator**

$$G_e(s) = \frac{\alpha(ZS + 1)}{(\alpha ZS + 1)}$$

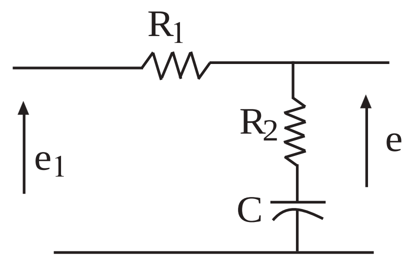
C_1 ; $\propto < 1$
ϕ = phase lead
$= \tan^{-1}(w\tau) - \tan^{-1}(\alpha w\tau)$



For maximum phase shift

w = Geometric mean of 2 corner frequencies

$$= \frac{1}{\tau\sqrt{\alpha}} \qquad \tan\phi_m = \frac{(1-\alpha)}{2\sqrt{\alpha}}$$



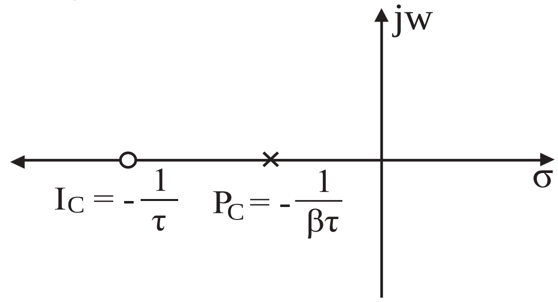
Effect

- * It increases Gain Crossover frequency
- * It reduces Bandwidth.
- * It reduces undamped frequency.

* Lag Compensators

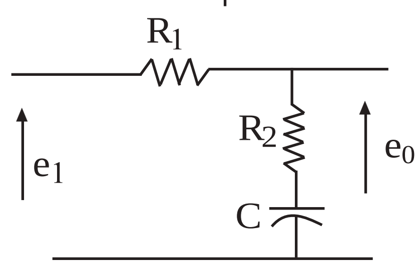
$$G_e(s) = \frac{\beta(1+\tau s)}{(1+\beta\tau s)} \; ; \; \beta > 1$$

$$G_e(jw) = \frac{(1+jw\tau)}{(1+jw\beta\tau)}$$



For maximum phase shift

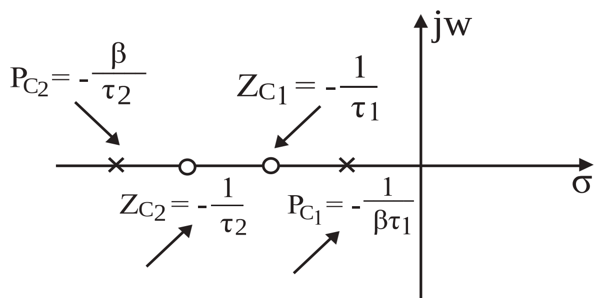
$$w = \frac{1}{\tau\sqrt{\beta}} \qquad \tan\phi_m = \frac{\beta - 1}{2\sqrt{\beta}}$$



Effect :

- * Increase gain of original Network without affecting stability.
- * Reduces steady state error.
- * Reduces speed of response.

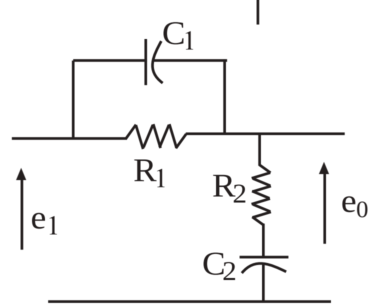
* Lag-Lead compensator



$$G_e(s) = \left(\frac{S + 1/\tau_1}{S + 1/\beta\tau_1}\right)\left(\frac{S + 1/\tau_2}{S + 1/\alpha\tau_2}\right)$$

$\beta > 1$; $\alpha < 1$

$$G_e(jw) = \frac{(1+jw\tau_1)(1+jw\tau_2)}{(1+jw\beta\tau_1)(1+jw\tau_2\alpha/\beta)}$$



State Variable Analysis

* The state of a dynamical system is a minimal set ot variables (known as state variables) such that the knowledge of these variables at $t = t_0$ together with the knowledge of input for $t \geq t_0$ completely determine the behaviour of system at $t > t_0$.

* **State variable**

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdots \\ x_n(t) \end{bmatrix} ; \; y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \cdots \\ y_p(t) \end{bmatrix} ; \; u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \cdots \\ u_m(t) \end{bmatrix}$$

Equations determining system behaviour :

$\dot{X}(t) = A\,x(t) + Bu(t)$; State equation

$Y(t) = C\,x(t) + Du(t)$; Output equation

State Transition Matrix

It is a matrix that satisfies the following linear homogeneous equation.

$$\frac{dx(t)}{dt} = Ax(t)$$

Assuming $\phi(t)$ is state transition matrix.

$$\phi(t) = ς^{-1}[(SI - A)^{-1}] \qquad \phi(t) = e^{At} = 1 + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \ldots\ldots$$

Properties :

1) $\phi(0) = 1$ (identify matrix)

2) $\phi^{-1}(t) = \phi(-t)$

3) $\phi(t_2 - t_1)\,\phi(t_1 - t_0) = \phi(t_2 - t_0)$

4) $[\phi(t)]^K = \phi(kt)$ for $K > 0$

Solution of state equation :

State Equation : $X(t) = A\,x(t) + Bu(t)$

$$X(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$

Relationship between state equations and Transfer Function

$$X(t) = Ax(t) + Bu(t)$$

Taking Laplace Transform both sides $\quad sX(s) = Ax(s) + Bu(s)$

$(SI - A)\,X(s) = Bu(s) \quad X(s) = (SI - A)^{-1}Bu(s)$

$y(t) = Cx(t) + Du(t)$

Take Laplace Transform both sides. $\quad y(s) = Cx(s) + Du(s)$

$x(s) = (SI - A)^{-1}Bu(s) \quad y(s) = [C(SI - A)^{-1}B + D]\,U(s)$

$$\frac{y(s)}{U(s)} = C(SI - A)^{-1}B + D = \text{Transfer function}$$

Eigen value of matrix A are the root of the characteristic equation of the system.

Characteristic equation $= |SI - A| = 0$

Controllability & Observability

* A system is said to be controllable if a system can be transferred from one state to another in specified finite time by control input u(t).

ENTRI

* A system is said to be completely observable if every state of system X_i (t) can be identified by observing the output y(t).

Test for controllability

Q_C = Controllability matrix = $[B \; AB \; A^2B \; ...... \; A^{n-1}B]$

Here A is assumed to be a n × n matrix

B is assumed to be a n × 1 matrix

If det (Q_C) = 0 system is uncontrollable

det (Q_C) ≠ 0, system is controllable

Test for observability

$$Q_0 = \text{observability matrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ . \\ . \\ . \\ C(A)^{n-1} \end{bmatrix}$$

A is a n × n matrix

C is (1 × n) matrix

If det (Q_0) = 0, System is unobservable

det (Q_0) ≠ 0, System is observable