DIFFERENTIAL EQUATION

An ordinary differential equations is that in which all the differential coefficients all with respect to a single independent variable. Thus the equation (a) to (d) are all ordinary differential equations. (e) is a system of ordinary differential equations.

A partial differential equations is that which there are two or more independent variables and partial differential coefficients with respect to any of them. The equations (f) and (g) are partial differential equations.

The order of a differential equation is the order of the highest derivative appearing in it. The degree of a differential equation is the degree of the highest derivative occurring in its, after the equation has been expressed in a form free from radicals and fractions as far as the derivatives are concerned.

Solution of a Differential Equation

A solution 9or integral) of a differential equation is a relation between the variable which satisfies the given differential equation.

For example, $y = ce^{\frac{x^3}{3}}$ (i)

is a solution of $\frac{dy}{dx} = x^2y$ (ii)

The general (or complete) solution of a differential equation is that in which the number of arbitrary constants is equal to the order of the differential equation. Thus (i) is a general solution of (ii) as the number of arbitrary constants (one constant c) is the same as the order of the equations (ii) (first order).

Similarly, in the general solution of a second order differential equation, there will be two arbitrary constants.

A particular solution is that which can be obtained from the general solution by giving particular values to the arbitrary constants.

For example, $y = 4e^{\frac{x^3}{3}}$

is a particular solution of the equation (ii), as it can be derived from the general solution (i) by putting c = 4.

A differential equation may sometimes have an additional solution which cannot be obtained from the general solution by assigning a particular value to the arbitrary constant. Such a solution is called a singular solution and usually is not of much practical interest in engineering.

Equations of the First Order and First Degree

It is not possible to analytically solve such equations in general. We shall, however discuss some special methods of solution which are applied to the following types of equations.

1. Equations where variables are separable.
2. Homogenous equations
3. Linear equations
4. Exact equations

In other cases, the particular solution may be determined numerically.

Variables Separable

If in an equation it is possible to collect all functions of x and dx on one side and all the functions of y and dy on the other side, then the variables are said to be separable. thus the general form of such an equation is $f(y)\,dy = \phi(x)\,dx$

Integrating both sides, we get $\int f(y)\,dy = \int \phi(x)\,dx + c$ as its solution.

Homogeneous Equations

Homogeneous equations are of the form $\frac{dy}{dx} = \frac{f(x, y)}{\phi(x, y)}$

where $f(x, y)$ and $\phi(x)$ $\phi(x, y)$ homogeneous functions of the same degree in x and y.

Homogeneous Function :

An expression of the form $a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \ldots\ldots + a_nY^n$ in which every term is of the nth degree, is called a homogeneous function of degree n. This can be rewritten as $x^n [a_0 + a_1 (y/x) + a_2 (y/x)^2 + \ldots.. + a_n (y/x)^n]$

Thus any functions $f(x, y)$ which can be expressed in the form $x^n f(y/x)$, is called a homogeneous function of degree n in x and y. For instance $x^3 \cos(y/x)$ is a homogeneous function of degree 3 in x and y.

To solve a homogeneous equation

1. Put $y = vx$, then $\frac{dy}{dx} = v + x\frac{dy}{dx}$
2. Separate the variables v and x, and integrate.

Leibnitze linear equation

The standard form of a linear equation of the first order, commonly known as Leibnitz's linear equation, is

$$\frac{dy}{dx} + Py = Q \text{ where P, Q are arbitrary functions of x} \qquad \text{(i)}$$

To solve the equation, multiply both sides by $e^{\int Pdx}$ so that we get

$$\frac{dy}{dx}.e^{\int Pdx} + y(e^{\int Pdx}P) = Qe^{\int Pdx} \text{ i.e. } \frac{d}{dx}(ye^{\int Pdx}) = Qe^{\int Pdx}$$

Integrating both sides, we get $ye^{\int Pdx} = \int Qe^{\int Pdx} dx + c$ as the required solution

Bernoulli's Equation

The equation $\frac{dy}{dx} + Py = Qy^n$ (i)

where P, Q are functions of x, is reducible to the Leibnitz's linear and is usually called the Bernoulli's equation

To solve (i), divide both sides by y^n, so that $y^{-n}\frac{dy}{dx} + Py^{1-n} = Q$ (ii)

Put $y^{1-n} = z$ so that $(1 - n)\, y^{-n}\frac{dy}{dx} = \frac{dz}{dx}$

$\therefore$ Eq. (ii) becomes $\frac{1}{1-n}\frac{dz}{dx} + Pz = Q$

or $\frac{dz}{dx} + P(1 - n)z = Q(1 - n)$,

which is Leibnitz's linear in z and can be solved easily.

Exact Differential Equations

1. Definition : A differential equation of the form $M(x, y)dx + N(x, y)dy = 0$ is said to be exact if its left hand member is the exact differential of some function $u(x, y)$ i.e. $du = Mdx + Ndy = 0$. Its solution, therefore, is $u(x, y) = c$.

2. Theorem : The necessary and sufficient condition for the differential equations $Mdx + Ndy = 0$ to be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

3. Method of solution : It can be shown that, the equation $Mdx + Ndy = 0$ becomes

$$d[u + \int f(y)dy] = 0$$

Integrating $u + \int f(y)\, dy = 0$

But $u = \int Mdx$ and $f(y)$ = terms of N not containing x

$\therefore$ The solution of $Mdx + Ndy = 0$ is

$\int Mdx + \int$ (terms of N not containing x) $dy = c$

(Provides of course that the equation is exact i.e. $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$)

Linear Differential Equations (Of n^{th} Order)

Definitions

Linear differential equations are those in which the dependent variable its derivatives occur only in the first degree and are not multiplied together. The general linear differential equation of the n^{th} order is of the form

$$\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1}y}{dx^{n-1}} + p_2 \frac{d^{n-2}y}{dx^{n-2}} + \dots + P_n Y = X$$

where $p_1, p_2, \dots, p_n$ and X are functions of x only

∴ The complete solution (C.S) of (iii) is y = C.F + P.I

Thus in order to solve the equation (iii), we have to first find the C.F i.e. the complementary function of (i) and then the P>I i.e. a particular solution of (iii)

Operator D Denoting $\frac{d}{dx}, \frac{d^2}{dx^2}, \frac{d^3}{dx^3}$ etc., so that

$\frac{dy}{dx} = Dy \; \frac{d^2y}{dx^2} = D^2y \; \frac{d^3x}{dx^3} = D^3y$ etc., the equation (iii) above can be written in the symbolic form.

$$(D^n + k_1 D^{n-1} + \dots + k_n)y = X$$

i.e. $\quad f(D)y = X$

where $f(D) = D^n + k_1 D^{n-1} + \dots + k_n$ i.e. a polynomial in D.

Thus the symbol D stands for the operation of differential and can be treated much the same as an algebric quantity i.e. f(D) can be factorised by ordinary rules of algebra and the factors may be taken in any order. For instance

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = (D^2 + 2D - 3)y$$
$$= (D + 3)(D - 1)y \text{ or } (D - 1)(D + 3)y$$

Rules for Finding the Coplementary Function

To solve the equation $\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + k_n y = 0$ ... (i)

where k's are constants.

The equation (i) in symbolic form is

$(D^n + k_1 D^{n-1} + k_2 D^{n-2} + \dots + k_n)\, y = 0$ .... (ii)

Its symbolic co-efficient equated to zero i.e

$D^n + k_1 D^{n-1} + k_2 D^{n-2} + \dots + k_n = 0$

is called the auxiliary equation (A.E) Let $m_1, m_2, \dots, m_n$ be its roots. Now 4 cases arise

Case I : If all the roots be real and different, Then (ii) is equivalent to

$(D - m_1)(D - m_2) \dots (D - m_n)\, y = 0$ ... (iii)

Now (iii) will be satisfied by the solution of $(D - m_n)\, y = 0$, i.e. by

$\frac{dy}{dx} - m_n y = 0$

This is a Leibnitz's linear and I.F = $e^{-m_n x}$

∴ Its solution is $ye^{-m_n x} = c_n$, i.e. $y = c_n e^{m_n x}$

Similarly, since the factors in (iii) can be taken in any order, it will be satisfied by the solutions of

$(D - m_1) y = 0$, $(D - m_2) = 0$ ets., i.e. by $y = c_1 e^{m_1 x}$. $y = c_2 e^{m_2 x}$ etc

Thus the complete solution of (i) is $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + ... + c_n e^{m_n x}$ .... (iv)

Case II : If two rootd are equal (i.e. $m_1 = m_2$), then (iv) becomes

$$y = (c_1 + c_2)e^{m_1 x} + c_3 e^{m_3 x} + ... + c_n e^{m_n x}$$

$$y = Ce^{m_1 x} c_3 e^{m_3 x} + ... + c_n e^{m_n x}$$

[∵ $c_1 + c_2$ = one arbitrary constant C]

Case III : If one pai of roots be imaginary, i.e

$$m_1 = \alpha + i\beta$$

$$m_2 = \alpha - i\beta$$

Then the complete solution is

$$y = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x} + c_3 e^{m_3 x} + ... + c_n e^{m_n x}$$

$$= e^{\alpha x}(c_1 e^{i\beta x} + c_2 e^{-i\beta x}) + c_3 e^{m_3 x} + ... + c_n e^{m_n x}$$

$$= e^{\alpha x}[c_1 (\cos \beta x + i \sin \beta x) + c_2 (\cos \beta x - i \sin \beta x)] + c_3 e^{m_3 x} + ... + c_n^{m_n x}$$

[∵ By Euler's Theorem, $e = \cos \theta + i \sin \theta$]

$$= e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) c_3 e^{m_3 x} + ... + c_n e^{m_n x}$$

where $C_1 = c_1 + c_2$

and $C_2 = i(c_1 - c_2)$

Rules For Finding The Particular Integral

Consider the equation $\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + ... + k_n y = X$

which in symbolic form is $(D^n + k_1 D^{n-1} + k_2 D^{n-2} + ... + k_n) y = X$

$$\therefore \text{P.I} = \frac{1}{D^n + k_1 D^{n-1} + k_2 D^{n-2} + ... + k_n}$$

Case I . When X = e^{ax}

Since $De^{ax} = ae^{ax}$

$D^2 e^{ax} = a^2 e^{ax}$

. . . .

. . . .

$D^n e^{ax} = a^n e^{ax}$

$$(D^n + k_1 D^{n-1} ... + k_n)e^{ax} = (a^n + k_1 a^{n-1} .... + k_n) e^{ax}$$

i.e. $f(D) e^{ax} = f(a) e^{ax}$

Operating on both sides by

$$\frac{1}{f(D)} . f(D)e^{ax} = \frac{1}{f(D)} . f(a)e^{ax}$$

or $$e^{ax} = f(a)\frac{1}{f(D)}e^{ax}$$

$\therefore$ by $\div f(a)$

$$\therefore \quad \frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax} \text{ provided } f(a) \neq 0 \qquad \text{(i)}$$

If $f(a) = 0$, the above rule fails and we proceed further.
It can be proved that in that case,

$$\frac{1}{f(D)}e^{ax} = x\frac{1}{f'(a)}e^{ax} \qquad \text{..... (ii)}$$

If $f'(a) = 0$, then applying (2) again, we get $\frac{1}{f(D)}e^{ax} = x^2\frac{1}{f''(a)}e^{ax}$ provided $f''(a) \neq 0$ .... (iii)

and so on.

Case II : When X = sin(ax + b) or cos (ax + b)

$$\frac{1}{f(D^2)}\sin(ax+b) = \frac{1}{f(-a^2)}\sin(ax+b) \text{ provided } f(-a^2) \neq 0$$

If $f(-a^2) = 0$, the above rule fails and we can prove that,

$$\frac{1}{f(D^2)}\sin(ax+b) = x\frac{1}{f'(-a^2)}\sin(ax+b) \text{ provided } f'(-a^2) \neq 0$$

If $f'(-a^2) = \frac{1}{f(D^2)} . \sin(ax+b) = x^2\frac{1}{f''(-a^2)}\sin(ax+b)$, provided $f''(-a^2) \neq 0$

and so on.

Similarly $$\frac{1}{f(D^2)}\cos(ax+b) = \frac{1}{f(-a^2)}\cos(ax+b), \text{ provided } f(-a^2) \neq 0$$

If $f(-a^2) = 0$, $\frac{1}{f(D^2)}\cos(ax+b) = x . \frac{1}{f'(-a^2)}\cos(ax+b)$, provided $f'(-a^2) \neq 0$

If $f'(-a^2) = 0$, $\frac{1}{f(D^2)}\cos(ax+b) = x^2 . \frac{1}{f''(-a^2)}\cos(ax+b)$, provided $f''(-a^2) \neq 0$

and so on ....

Case III : When X = x^m

Here $$P.I = \frac{1}{f(D)}x^m = [f(D)]^{-1}x^m$$

Expand $[f(D)]^{-1}$ in ascending powers of D as far as the term in D^m and operate on x^m term by term. Since the $(m+1)^{th}$ and higher derivatives of x^m are zero, we need not consider terms beyond D^m

When $X = e^{ax} V$, where V is a function of x.

$$P.I = \frac{1}{f(D)}e^{ax} V = e^{ax}\frac{1}{f(D+a)}V$$