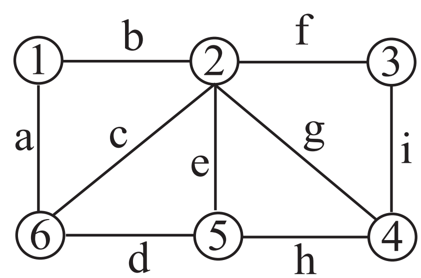


WALK

An alternating sequence of vertices and edges, that begin and ends with a vertex.

Ex : 1b2e5h4

Trail: A trail is a walk without repeated edges.

Ex: 1b2f3i4h5e2g4

Path: A path is a walk without repeated vertices.

1b2f3i4h5

Closed walk: A walk if the first and last vertices are same.

1b2e5d6a1

Open walk: A walk if the first and last vertices are different.

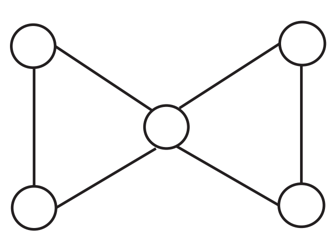
Circuit or Cycle: A circuit is a path which ends at the vertices it begins.

1b2e5d6a1

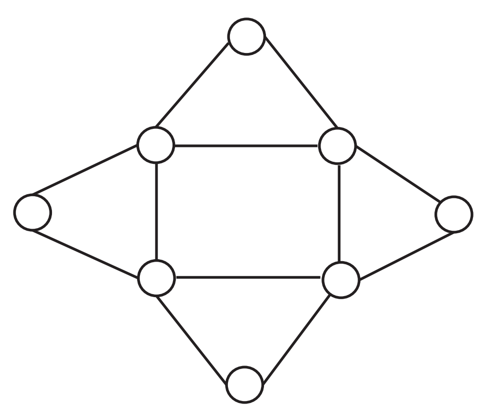
NOTE: By default walk is open walk.

Euler Graph:

A graph containing all the edges and no edges is repeated and having Closed walk is called Euler graph.

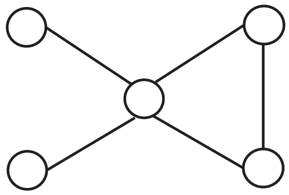


Euler graph



Euler graph

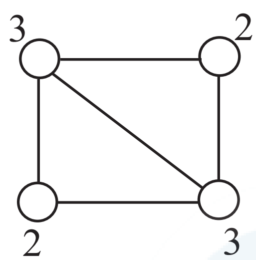
Result: A connected graph is Euler graph if degree of every vertex is even.



Universal graph (Not Euler graph)

An open walk containing all the edges of the graph and no esge is repeated called Universal graph.

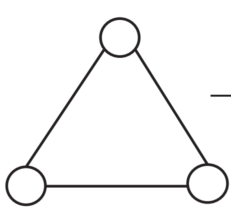
Result: A connected graph is called Universal Graph if there are two exactly two vertices of off degree.



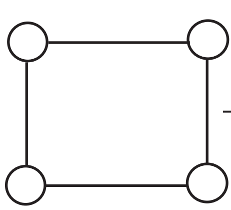
Universal Graph

- If H-cycle exist then H-path should be preset.
- If G is a connected Homiltonial graph with n vertices.
  1. No. of vertices in Hamiltonian cycle = n
  2. No. of edges in Hamiltonian cycle = n
  3. No. of vertices in Hamiltonian path = n
  4. No. of edges in Hamiltonian path = n - 1
  5. The degree of any vertex in H-cycle = 2

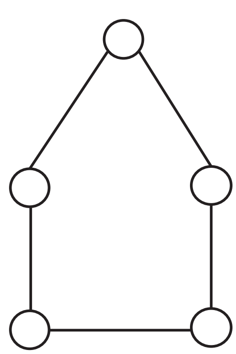
Simple Graph



→ 3 cycle



→ 4 cycle



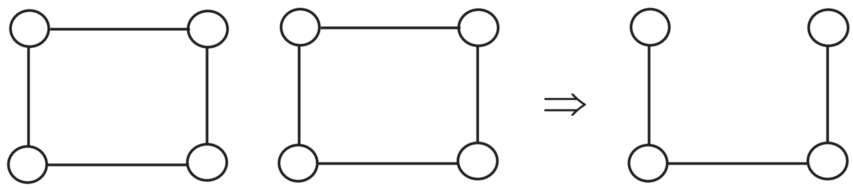
→ 5 cycle

Simple cycle graph with n vertices

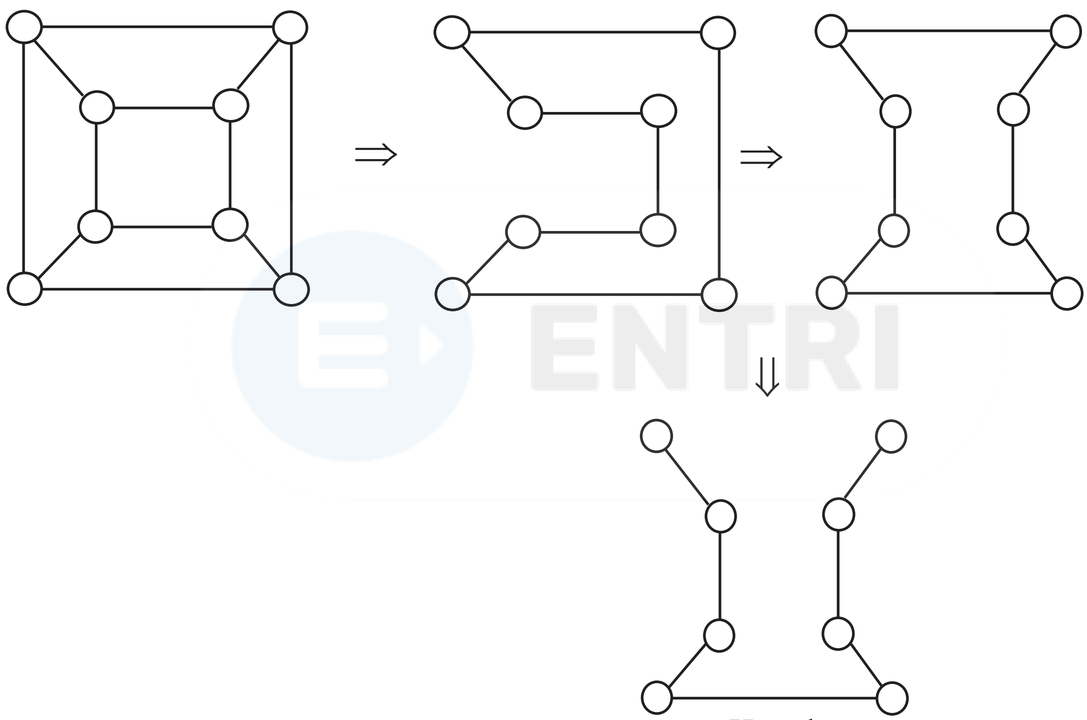
	V	e	d(v)
C_n	n	n	2

Hamiltonian Graph

A circuit containing all the vertices and no vertex is repeated except starting and ending.



Graph   Hamiltonian cycle   H-Path

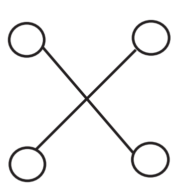


H-path

H-Path (Hamiltonian Path) : A path containing all the vertices and no vertices is repeated.

Cycles of length 3, 5, 7, 9, __________ odd cycles
Cycles of length 4, 6, 8, 10, _________ even cycles

Planner Graph

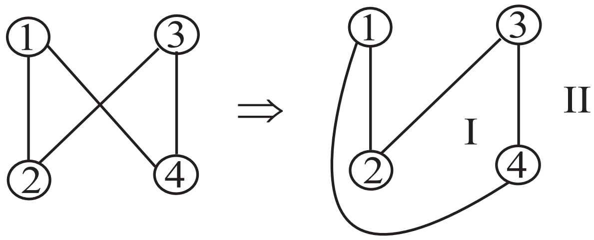


Cross over

Planar representation of a graph:

Drawing a graph in a plane without crossing.

A graph having planar representation in a plane is called Planar graph.



Planar Graph

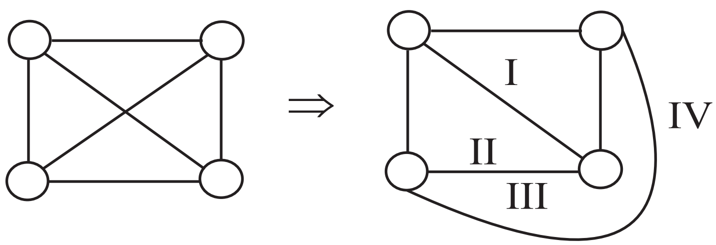
$\boxed{v - e + r = 2}$

$v = 4$

$e = 4$

$4 - 4 + 2 = 2$

$\underline{2 = 2}$



$v = 4$

$E = 6$

$r = 4$

$\boxed{4 - 6 + 4 = 2}$

The planar representation of planar graph divided entire plane lato regions or faces.

Degree of a region

The number of edges in the boundary of a region is called its degree.

Region	Degree
1	3
2	3
3	3
4	3

IVth region → Exterior region or unbounded region

Other region are Interior or bounded region.

Euler Formula

In any connected planar graph G with

V Vertices

E Edges

r Regions

We have $\boxed{v - e + r = 2}$

1. The sum of degrees of regions = twice the number edges.

$$\boxed{\sum \deg (r_i) = 2|E|}$$

2. Simple planar graph, minimum degree of region = 3

$$\sum \deg(r) = 2e$$

$$3 + 3 + 3 + 3 \text{ -------- } + 3 \leq 2e$$

$$\boxed{3r \leq 2e}$$

3. $2 = v - e + r$

$$2 \leq v - e + \frac{2e}{3}$$

$$2 \leq v - \frac{e}{3}$$

$$2 \leq \frac{3v - e}{3}$$

$$6 \leq 3v - e \qquad \boxed{e \leq 3v - 6}$$

In a simple connected planar graph with minimum degree of region = 3 → (assume in problem)

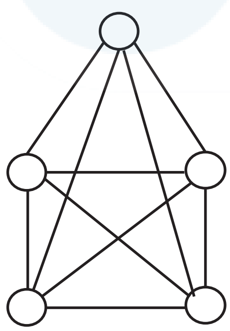
For planarity check $v - e + r = 2$

$3r \leq 2e$

$e \leq 3v - 6$

Q. k_5 is planar or not?

k_5



$v = 5$

$$E = 10 \left(\frac{n(n-1)}{2}\right)$$

1. $v - e + r = 2$

$5 - 10 + r = 2$

$r = 7$

2. $3r \leq 2e$

$3r \leq 2 \times 10$

$21 \leq 21$ → Not Possible

3. $e \leq 3v - 6$

$10 \leq 3 \times 5 - 6$

$10 \leq 9$ → Not possible

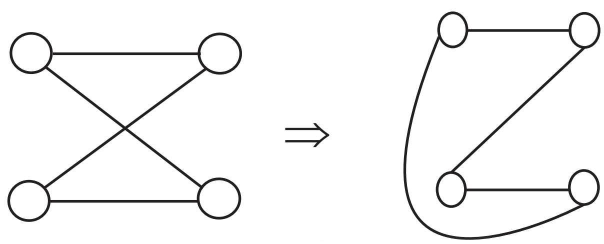
So k_5 is non-planar graph.

NOTE:
In a simple connected planar graph with minimum degree of region = k → any then results

$v - e + r = 2$

$kr \leq 2e$

$$\boxed{e \leq \frac{k(v-2)}{(k-2)}}$$



Minimum degree of region = 4

NOTE: In any Bi-particle graph minimum degree of region = 4

Q. $k_{3,3}$ is planar or not?

$k_{3,3}$ contain

$v = 6;\ E = 9$

1. $v - e + r = 2$

   $6 - 9 + r = 2 \quad \Rightarrow \quad \boxed{r = 5}$

2. Minimum degree of region = 4

   $4r \leq 2e$

   $4 \times 5 \leq 2 \times 9 \quad \Rightarrow \quad 20 \leq 18 \quad \rightarrow$ Not possible

3. $e \leq \dfrac{k(v-2)}{(k-2)}$

   $e \leq \dfrac{4(6-2)}{(4-2)} \quad \Rightarrow \quad e \leq \dfrac{4 \times 4}{2}$

   $9 \leq 8 \quad \rightarrow$ Not possible

So $k_{3,3}$ is non planar graph.

NOTE:

k_5 and $k_{3,3}$ both are known as Kuratowski's graph.

k_5 and $k_{3,3}$

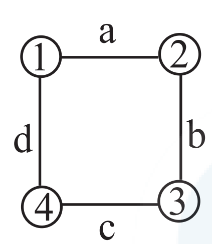
1. Both are non-planar.
2. Both are regular
3. Both gives a planar graph if an edge or a vertex is removed.
4. k_5 is a non-planar graph with smallest number of vertices.
5. $k_{3,3}$ is a non-planar graph with smallest number of edges.

Kuratowski's Result

A graph G is planar if it does not contain any graph Homeomarphic to k_5 or $k_{3,3}$.

Matching:-

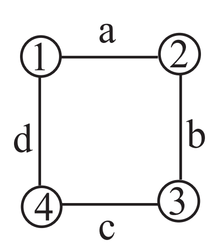
The set of non-adjacent edges.



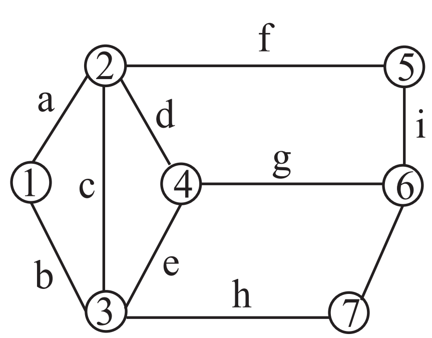
$M_1 = \{a\}$
$M_2 = \{a, c\}$
$M_3 = \{b, d\}$

Matching number → ($\alpha'(G)$)

Maximum no. of non-adjacent is called Matching number.



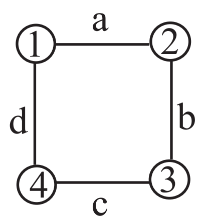
$\alpha'(G) = 2$

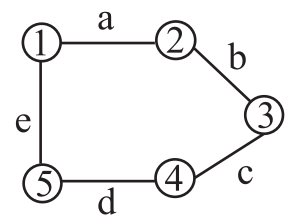


$\{f, g, h\}$
$\{a, i, h\}$
$\alpha'(G) = 3$

Edge Covering

The set of edges which can cover all the vertices of positive degree.





$E_1 = \{a, b, c, d\}$

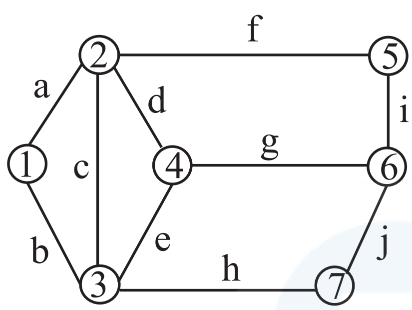
$E_2 = \{a, c\}$

$E_3 = \{b, d\}$

$E_1 = \{e, b, c\} + 1$

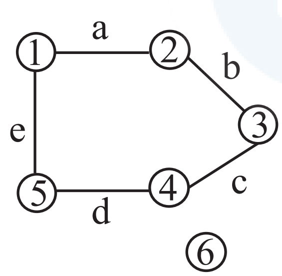
Edge Covering Number (β'(G)):-

Minimum number of edges which can cover all the vertices of positive degree + number of isolated vertices (If any)



$E_1 = \{a, i, h, e\}$

$\beta'(G) = 4$



$E = \{e, b, c\} + 1$

$\beta'(G) = 4$

NOTE :

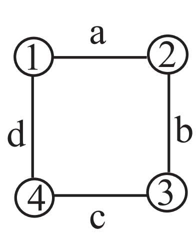
In a simple graph with 20 vertices the matching number = 8, then edge covering no.

(a) 10 (b) 12 (c) 14 (d) 20

In a simple graph with n vertices matching no. $(\alpha'(G)$ + edge covering number $(\beta'(G))$ = n

$8 + n = 20 \quad \rightarrow \quad n = 12$

Independent set → Set of Non-adjacent vertices



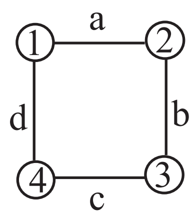
$V_1 = \{1\}$

$V_2 = \{1, 3\}$

$V_3 = \{2, 4\}$

Independance number (α(G)) :-

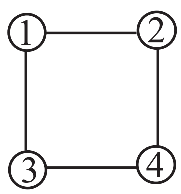
The maximum no. of non-adjacent vertices.



$\alpha(G) = 2$

Vertex Covering

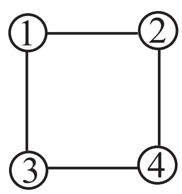
The set of vertices which can cover all the edges.



$V_1 = \{1, 4\}$
$V_2 = \{2, 3\}$
$V_3 = \{1, 2, 3, 4\}$

Vertex Covering Number → (β(G))

Minimum number of vertices which can cover all the edges.

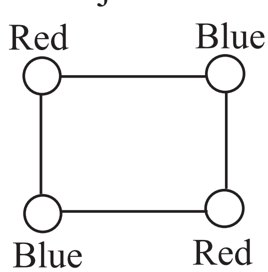


$\beta(G) = 2$

A simple graph with n vertices $\alpha(G) + \beta(G) = n$

Graph Coloring Problem

Coloring the vertices of the graph such that adjacent vertices have different color (or) no. of two adjacent vertices having same color.



Chromatic Number (χ(G)) :-

Minimum number of colors required to color the graph.

Graph (G)	χ(G)	
N_n (Null graph)	1	
C_n	2	Even cycle
$(n \geq 3)$	3	Odd cycle
K_n (Complete graph)	n	
$K_{m,n}$ (Bipartite graph)	2	

The following statements are equivalent

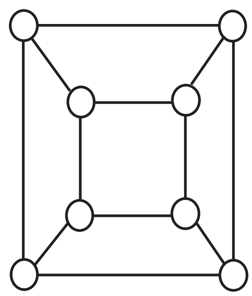
(1) G is bi-partite

(2) G is 2-colorable ($\chi(G) = 2$)

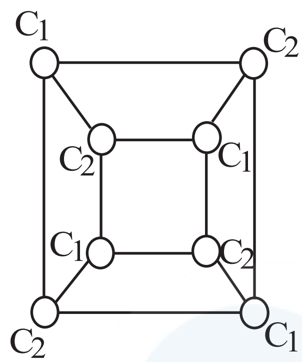
(3) Every cycle in G is even cycle

Ex:

GATE : 2004



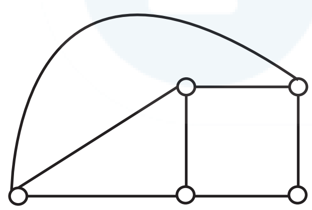
Solution



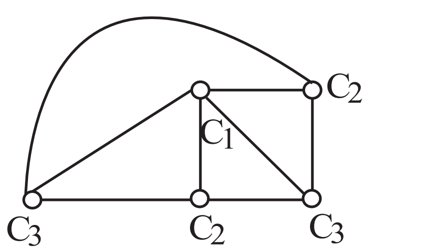
$\chi(G) = 2$

Ex:

GATE : 2001

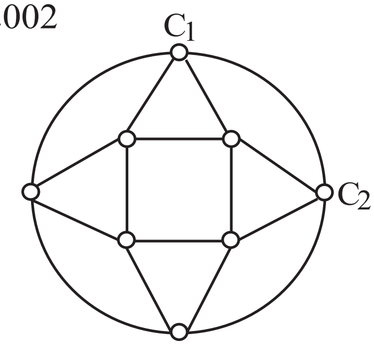


Solution



$\chi(G) = 3$

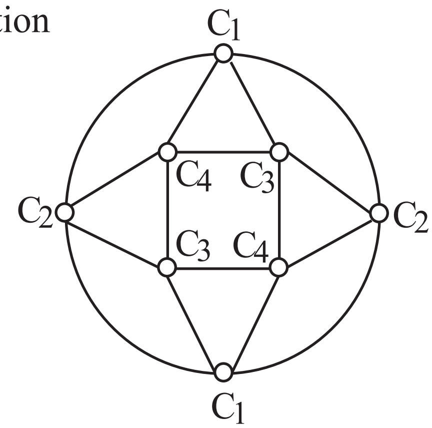
GATE : 2002



(a) 3 (b) 4

(c) 5 (d) 6

Solution



Answer = B

TREE

A tree is a connected acyclic graph i.e. connected and having number cycle.

- The following statements are equivalent
  1. Connected and acyclic graph.
  2. Connected and has (n - 1) edges.
  3. Acyclic and has (n - 1) edges.
  4. There is exactly one path between any two vertices.
  5. Minimally connected.

EX: T is a tree with: 4 vertices of degree 2; 2 vertices of degree 3; and remaining vertices of degree 1. How many vertices of degree 1 are there?

Solution:

4 vertices of degree 2 = $4 \times 2 = 8$

2 vertices of degree 3 = $2 \times 3 = 6$

x vertices of degree 1 = $x \times 1 = x$

$6 + x$ $\qquad$ $x + 14$

Number of edges = $(6 + x) - 1 = 5 + x$

Sum of degree = 2e

$14 + x = 2(5 + x)$

$x = 4$ - Number of vertices of degree 1

Q. T is a tree with: 6 vertices of degree 2; 3 vertices of degree 3; and remaining vertices of degree 1

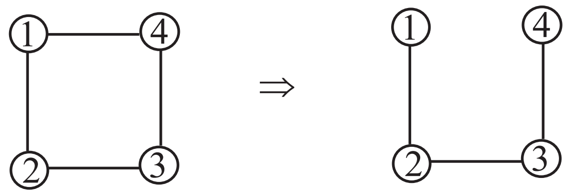
(1) How many vertices of degree 1 are there?

(2) How many vertices are there?

Solution: 5 Vertices

NOTE: Every tree is Bi-partite ($n \geq 2$)

Spanning Tree

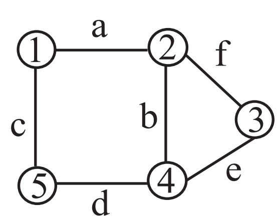


The spanning tree of connected simple graph is a spanning subgraph which is a tree.

Construction of Spanning Tree

DFS (Depth First Search)

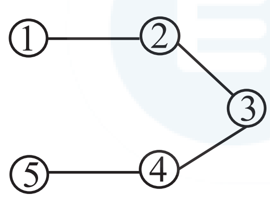
At a given opportunity we go to next higher level and back track, if needed.



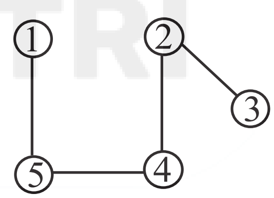
Q. Which of the following sequence of vertices are not traversed by DFS?

(A)	1	2	3	4	5	(B)	1	5	4	2	3
(C)	1	2	4	5	3	(D)	1	2	5	4	3

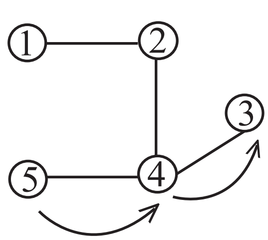
(A)



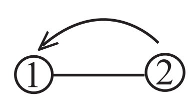
(B)



(C)



(D)



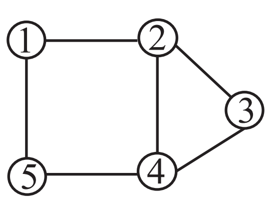
Unnecessary back tracking

We avoid Un-necessary backtracking in DFS

BFS (Breadth First Search)

At a given opportunity complete the level and then move to next level.

Q.



Which are not possible using BFS?

(A)	1	2	5	3	4
(B)	1	2	5	4	3
(C)	1	5	2	4	3
(D)	1	2	4	5	3

(A) | 1 |

Delete '1' and explore its children

| 2 | 5 |

Delete '2' and explore its children

| 5 | 3 | 4 |

Delete '5' and explore its children

| 3 | 4 |

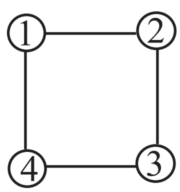
Delete '3' and explore its children

ADJACENCY GRAPH

Let G = (V, E) be a simple graph with n vertices

the adjacency matric of G $\quad A_G = [m_{ij}]_{n\times n}$

$$m_{ij} = \begin{cases} 0 & \text{If edge } \{i, j\} \quad E \\ 1 & \text{If edge } \{i, j\} \quad E \end{cases}$$



$$A_G = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

i, j^{th} entry in A_G gives the number of paths of length '1' from vertex i to vertex 'j'.

$A_G^2 = A \cdot A$

$$A_G^2 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$A_G^2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 3 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 3 & 1 \\ 1 & 2 & 1 & 2 \end{bmatrix} \end{array}$$

$A_G^2 \rightarrow$ the i, j^{th} entry in A_G^2 gives the number of paths of length 2 between i and j.

$A_G^3 \rightarrow$ the i, j^{th} entry in A_G^3 gives the number of paths of length 3 between i and j.

$$\left.\begin{array}{l} 1 - 2 - 1 \\ 1 - 3 - 1 \\ 1 - 4 - 1 \end{array}\right\} \text{3 path} \qquad \left.\begin{array}{l} 1 - 2 - 3 \\ 1 - 4 - 3 \end{array}\right\} \text{2 path}$$

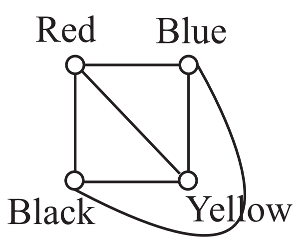
Graph Theory Problems

GATE : 2016

Q. The minimum number of colours that is sufficient to vertex-colour any planar graph is _______?

Solution:

$k_4 \rightarrow$

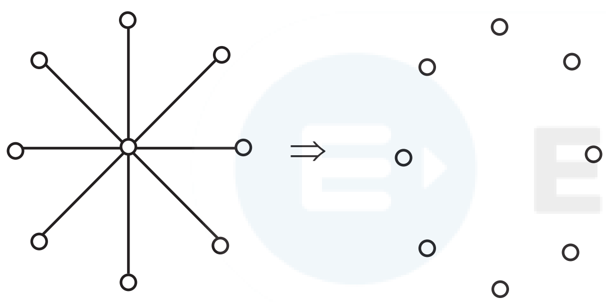


Every planar graph can color with 4 colors that means four colours are sufficient to properly color any planar graph.

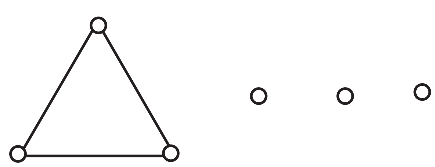
GATE : 2003

Let G be an arbitrary graph with n nodes and k components. If a vertex is removed from G, the number of component in the resultant graph must necessarily lie between

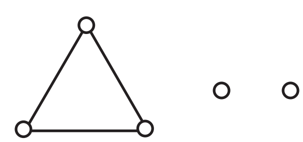
(a) k and n (b) k - 1 and k + 1 (c) k - 1 and n - 1 (d) k + 1 and n - k



k = 1 **k = 8**



k = 4



k = 3

[k - 1, n - 1]

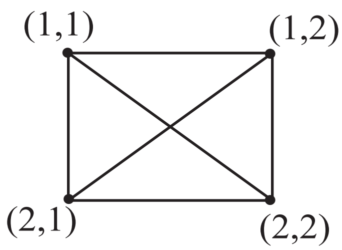
Q. Consider an undirected random graph of eight vertices. The probability that there is an edge between a pair of vertices is ½. What is the expected number of un-ordered cycles of length three?

$$= \Sigma x \, p(x)$$

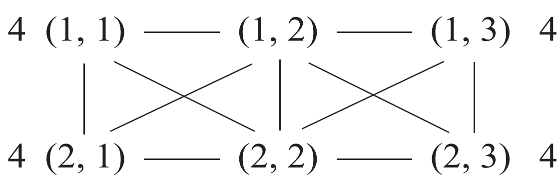
$$= 8c_3 \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = 7$$

Q. Consider an undirected graph G where self loops are not allowed. The vertex set of G is $\{(i, j) : 1 \leq i \leq 12,\ 1 \leq j \leq 12\}$.
There is an edge between (a, b) and (c, d)
if $|a - c| \leq 1$ and $|b - 2| \leq 1$
The number of edges in this graph is ____?

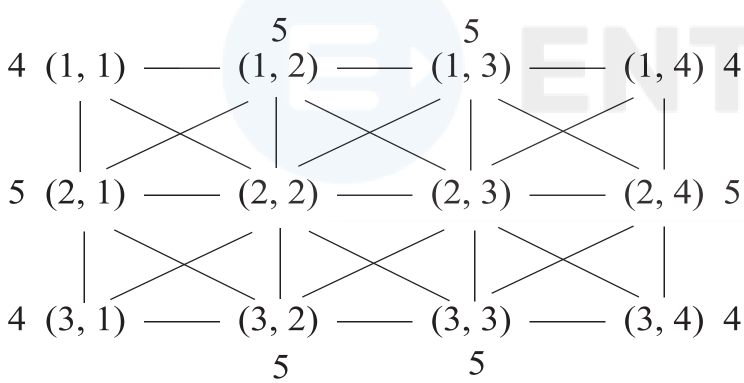
Solution



$4 \times 3 = 2E$
$E = 6$

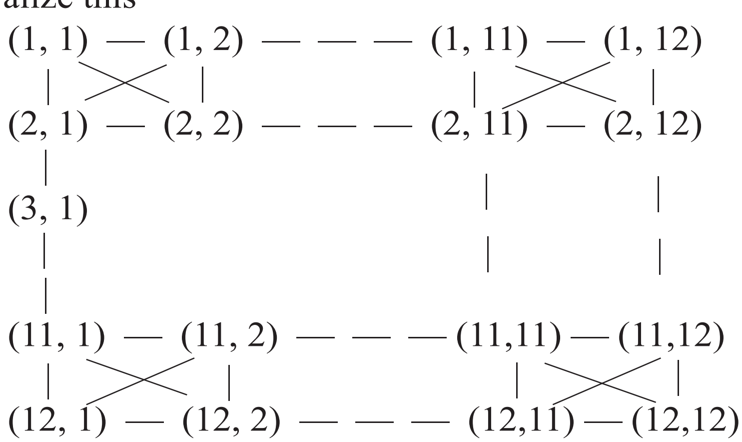


$4 \times 3 + 2 \times 5 = 2E \Rightarrow 12 + 10 = 2E \Rightarrow E = 11$



$\underline{4} \times 3 + \underline{6} \times 5 + \underline{2} \times 8 = 2E$
$\Rightarrow 12 + 30 + 16 = 2E \Rightarrow 42 + 16 = 2E \Rightarrow 58 = 2E \Rightarrow E = 29$

Generalize this



From above diagram

(1) The four corner vertices have each 3 degrees which gives $4 \times 3 = 12$ degrees.

(2) The 40 side vertices have 5 degrees each contributing a total of $40 \times 5 = 200$ degrees.

(3) The 100 interior vertices each have 8 degrees contributing a total $100 \times 8 = 800$ degrees

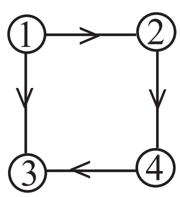
50 total degrees of the graph

$12 + 200 + 800 = 1012$ degree

$1012 = 2E$

$E = 500$

Directed Graph (Di-Graph)



$G = (V, E)$

V = Vertex set $\{V_1, V_2, V_3, \text{------}, V_n)$

E = Edge set $\{E_1, E_2, E_3, \text{--------}, E_n)$

Indegree : The number of edges incident into the vertex.

Outdegree: The number of incident out of the vertex.

First theorem of the directed graph : -

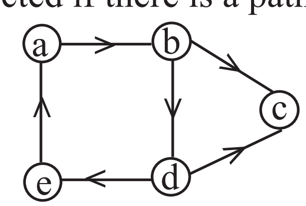
In a directed graph

Vertex	In	Out
1	0	2
2	1	1
3	2	0
4	1	1
	4	4

The sum of indegree is = the sum of outdegree = the number of edges in the graph

Strongly Connected

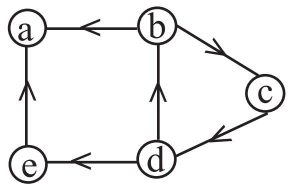
A directed graph is strongly connected if there is a path from a to b and from b to a where a and b are vertices in the graph.



Strongly connected because there is a path between any two vertices in this directed graph.

Weakly Connected : -

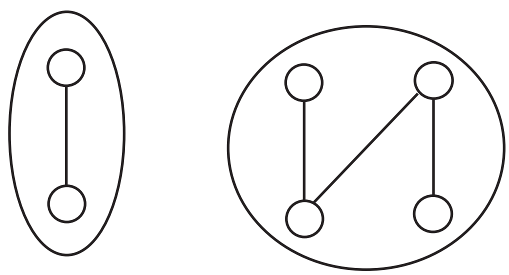
A directed graph is weakly connected if there is a path between every two vertices in the underlying undirected graph.



not strongly connected
there is no direct path from a to b in this graph.
It is weekly connected.

FOREST:

- A forest is an undirected acyclic graph
- A forest is an undirected graph, all of whose connected components are trees.
- The graph consists of a disjoint union of trees.



k = 2

GATE : 2014

If G is a forest with n vertices and k connected components, how many edges does G have?

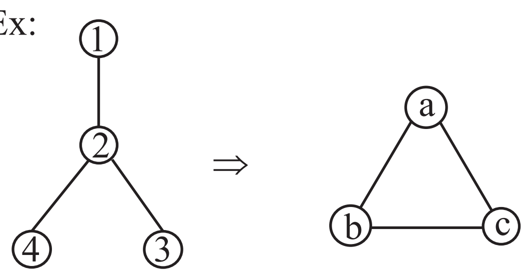
(a) $\left\lfloor \frac{n}{k} \right\rfloor$ (b) $\left\lceil \frac{n}{k} \right\rceil$ (c) n - k (d) n - k + 1

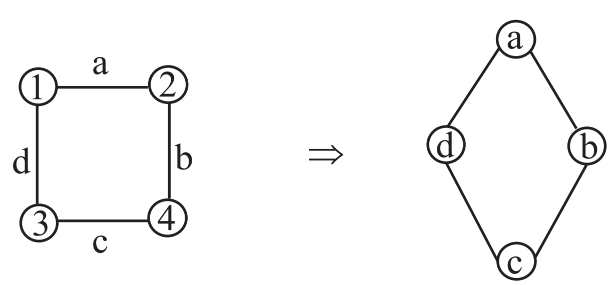
Line Graph L(G) : -

The line graph L(G) of graph H is constructed as follows:

1. For every edge in G there is a vertex in L(G).
2. Two vertices in L(G) are adjacent if their corresponding edge in G are adjacent.

Ex:

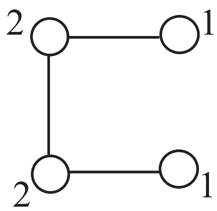




→ line graph of a cycle is cycle.

NOTE:

Minimum degree δ;   Maximum degree Δ



$\delta = 1$

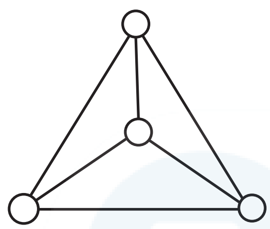
$\Delta = 2$

Result: In any graph G = (V, E) with V - vertices, E - Edges.

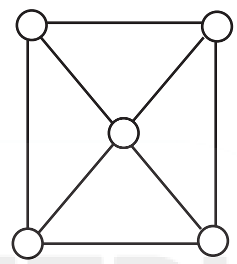
$$\delta \leq \frac{2e}{V} \leq \Delta$$

WHEEL $(n \geq 3)$

When we add an additional vertex to the cycle $(C_n, n \geq 3)$ and connect this new vertex to each of the n vertices in G, by new edges.



w_3



w_4

Q. A connected planar simple graph has 20 vertices each of degree 3. How many regions does a representation of this planar graph split the plane?

Solution:

$\sum \deg(V) = 2|E|$

$\Rightarrow \quad 20 \times 3 = 2E \quad \Rightarrow \quad E = 30$

$V - E + r = 2$

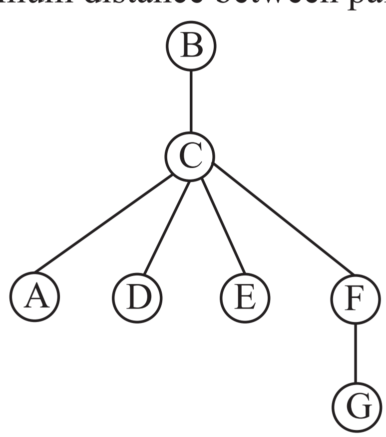
$\Rightarrow \quad 20 - 30 + r = 2 \quad \Rightarrow \quad r = 12$

Diameter of Graph

The diameter of graph is the maximum distance between pair of vertices.

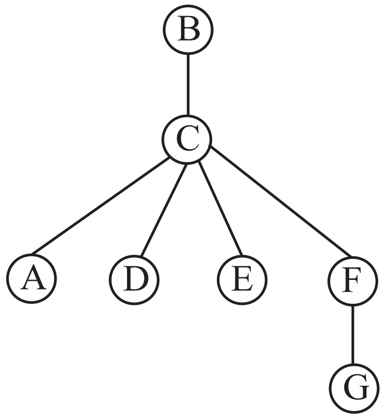
Diameter = 3

$BC \rightarrow CF \rightarrow FG$



Radius of Graph

The minimum among all the maximum distance between a vertex to all other vertices.



Radius = 2

$BC \rightarrow CF$

$BC \rightarrow CE$

$BC \rightarrow CD$

$BC \rightarrow CA$

GATE : 2015

Let G be a connected planar graph with 10 vertices. If the number of edges on each face is three, then the number of edges in G is ________?

Answer:

Number of vertices = 10; $\quad d(r_i) = 3$

Number of edges = ?; $\quad (e) = ?$

$V - e + r = 2$

$10 - e + r = 2 \quad \Rightarrow \quad r = e - 8 \quad$ —— (1)

$\sum d(r_i) = 2e$

$3r = 2e \quad \Rightarrow \quad r = \frac{2e}{3} \quad$ —— (2)

Put r value in (1)

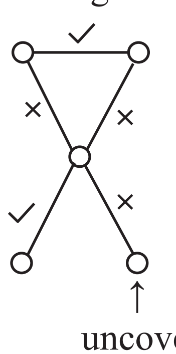
$\frac{2e}{3} = e - 8 \quad \Rightarrow \quad e = 24$

Alternate: $\quad e \leq 3V - 6$

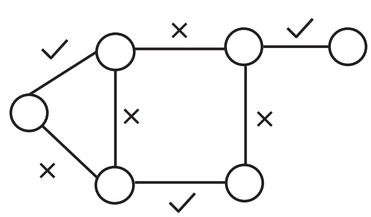
$e \leq 3 \times 10 - 6 \quad \Rightarrow \quad 6 \leq 24$

Perfect matching:-

It is matching with some special property and it cover all the vertices.



Matching

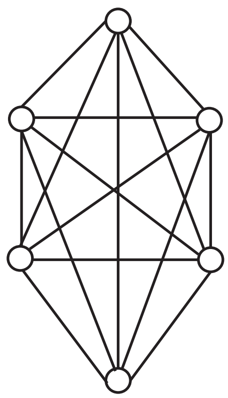


Perfect matching

GATE : 2004

Q. How many perfect matching are there in a complete graph of 6 vertices?

(a) 15 (b) 24 (c) 30 (d) 60



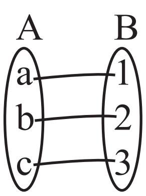
Number of Perfect matching
$= 5 \times 3 \times 1 = 15$

The number of perfect matching in complete graph
(n - 1) (n - 3) (n - 5) (n - 7) --------
where n is error.

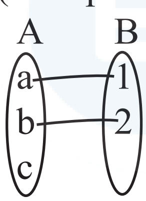
Functions

A function f from set A to set B is a rule which assign every element of set A to a unique element of B.

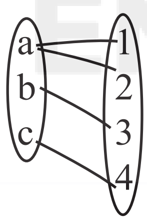
$F : A \rightarrow B$ (F maps A to B)



Function



Not a function

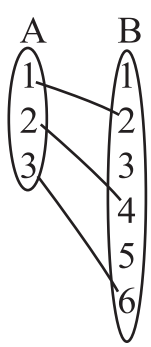


Not a function

$f : A \rightarrow B$ Function

$A = \{1, 2, 3\}$ $B = \{1, 2, 3, 4, 5, 6\}$

$f(x) = 2x$



Ex:- $f : 2^+ \rightarrow 2^+$ (2^+ = set of positive integers)

$f(x) = x - 3$; Is a function?

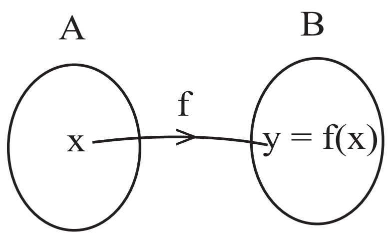
$f(1) = 1 - 3 = -2 \& 2^+$

So it is not a function.

Ex:- F : 2 → 2 (2 = set of integers)

f(x) = x - 3

Yes, it is a function.



f : A → B

A → Domain of the function; B → Co-domain of the function

y = f(x) - Image of x under f.

y = f(x), than x is called Preimage of y.

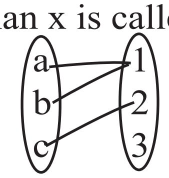


Image of (a) = 1

Image of (b) = 1

Image of (c) = 2

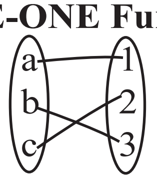
Preimage of (1) = {a, b} Preimage of (2) = {c}

Preimage of (3) = Not Preimage

Domain = {a, b,c} Co-domain = {1, 2, 3, 4} Range = {1, 2}

Range : The range of f is the set of all images of elements of A

ONE-ONE Funcion (Injective function)



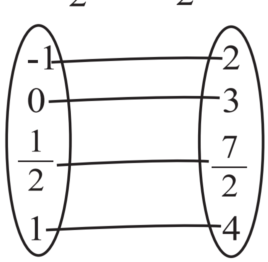
If	a ≠ b	then	f(a) ≠ f(b)
If	b ≠ c	then	f(b) ≠ f(c)
If	f(a) = f(b)	then	a = b

Q. f : R → R (R = set of real numbers)

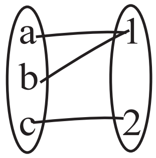
f(x) = x + 3 f is one-one

f(1) = 1 + 3 = 4; f(0) = 0 + 3 = 3; f(-1) = -1 + 3 = 2

$$f\left(\frac{1}{2}\right) = \frac{1}{2} + 3 = \frac{7}{2}$$

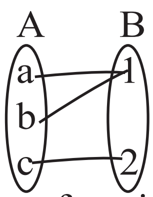


One-One

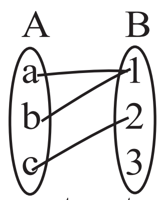


$a \neq b$ but $f(a) = f(b)$
So not one-one

ONTO Function (Surjective Function)



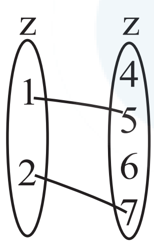
onto function



not onto

In onto every element has preimage from B to A.

Ex: $f : z \rightarrow z$ (z = set of integers)
$f(x) = 2x + 3$, then f is
(a) only one-one (b) only onto (c) both one-one and onto
(d) none



4, 6, 8 } does not have any preimage. So, not onto.

$f(1) = 2 \times 1 + 3 = 5$ $f(2) = 2 \times 2 + 3 = 7$

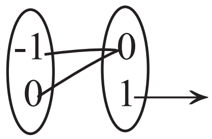
Only one-one
$y = 2x + 3$ (y in terms of x)
$2x + 3 = y \Rightarrow 2x - y - 3 \Rightarrow x = \frac{y - 3}{2}$ (x in terms of y)

Put y = 2
$$x = \frac{2 - 3}{2} = \frac{-1}{2}$$ and 2. So, not onto

GATE : 2004

Q. Consider the mapping function f_1 and f_2 as described below
$f_1 : R \rightarrow R$ defined by $f(x) = x^2 + x$, x and R
$f_2 : z \rightarrow z$ $f(y) = z^y$, y and z
R : set of real numbers; z : set of integers
then what about f_1 and f_2?

$\underline{f_1}$:



does not have any preimage

$f(-1) = (-1)^2 + (-1) = 1 - 1 = 0;$ $\qquad f(0) = (0)^2 + 0 = 0$

Not one-one

So, not onto.

$\underline{f_2}$:

$f(-1) = 2^{-1} = \frac{1}{2}$ and 2. It is not a function.

GATE : 2005

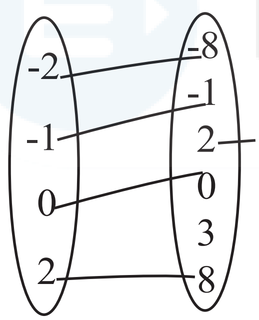
Q. Consider the following functions from 2 to 2, where 2 is the set of integers.

s_1 : $f(x) = x^3$ is one to one but not onto.

s_2 : $f(n) = \left\lceil \frac{n}{2} \right\rceil$ is onto but not one-one.

(a) s_1 only (b) s_2 only (c) both (d) none

Solution: $s_1 = f(x) = x^3$

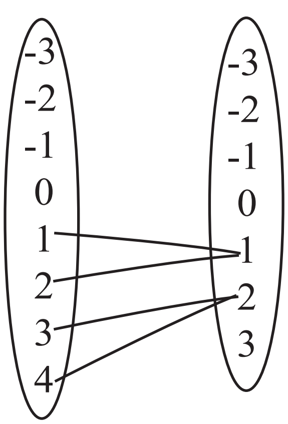


but 2 has no mapping. So not onto.

one-one

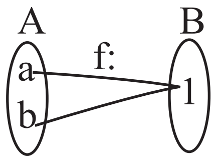
s_1 is true

$\underline{s_2}$



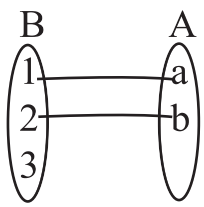
onto but not one-one

Q. Find the inverse of the following functions

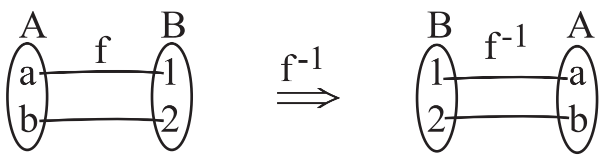


onto

$\Downarrow f^{-1}$



not a function



one-one and onto   inverse function

Rule : f^{-1} exist if f is one-one and onto.

Q. 1. $f(x) = x + 3$ find $f^{-1}(x)$?

$y = x + 3$ (y in terms of x)

$x = y - 3$ (x in terms of y)

$f^{-1}(y) = y - 3$

$f^{-1}(t) = t - 3$

$f^{-1}(x) = x - 3$

2. $f(x) = 2x + 3$

$y = 2x + 3 \quad \Rightarrow \quad x = \frac{y - 3}{2}$

$f^{-1}(x) = \frac{y - 3}{2} \qquad f^{-1}(x) = \frac{x - 3}{2}$

GATE : 2004

3. $f(x) = \frac{2x + 3}{x + 4} \qquad y = \frac{2x + 3}{x + 4}$

$xy + 4y = 2x + 3 \quad \Rightarrow \quad xy - 2x = 3 - 4y \quad \Rightarrow \quad x = \frac{3 - 4y}{y - 2}$

$$f^{-1}(y) = \frac{3 - 4y}{y - 2} \quad \Rightarrow \quad f^{-1}(x) = \frac{3 - 4x}{x - 2}$$

Q. GATE : 2005

$f : R \times R \rightarrow R \times R$

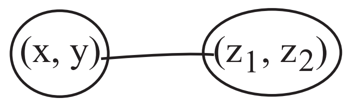
$f(x, y) = (x + y, x - y);$ $\quad f \quad (x, y) = ?$

Solution:

$f(x, y) = (x + y, x - y)$

$(z_1, z_2) = (x + y, x - y)$

$$x + y = z_1 \quad \text{——— (I)}$$
$$x - y = z_2 \quad \text{——— (II)}$$
$$2x = z_1 + z_2$$



(z_1, z_2) in terms of x and y

$$x = \frac{z_1 + z_2}{2}$$

put x in equation in (I)

$$\frac{z_1 + z_2}{2} + y = z$$

$$y = z_1 - \frac{z_1 + z_2}{2} = \frac{2z_1 - z_1 - 2z_2}{2} = \frac{z_1 - z_2}{2}$$

$$(x, y) = \left(\frac{z_1 + z_2}{2}, \frac{z_1 - z_2}{2}\right)$$

$$f^{-1}(z_1, z_2) = \left(\frac{z_1 + z_2}{2}, \frac{z_1 - z_2}{2}\right); \quad f^{-1}(x, y) = \left(\frac{x + y}{2}, \frac{x - y}{2}\right)$$

Ex:- $f : R \times R \rightarrow R \times R$

$f(x, y) = (2x + y, x - 2y);$ $\quad f^{-1}(x, y) = ?$

Solution:

$$2x + y = z_1 \quad \times 1 \quad \text{———— (1)}$$
$$x - 2y = z_2 \quad \times 2 \quad \text{———— (2)}$$

$$2x + y = z_1$$
$$2x - 4y = 2z_2$$
$$5y = z_1 - 2z_2 \quad \Rightarrow \quad y = \frac{z_1 - 2z_2}{5}$$

put y in anyone $\quad x = \dfrac{2x + y}{5}$

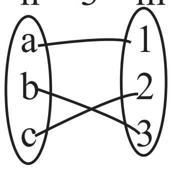
$$f^{-1}(x, y) = \left(\frac{2x + y}{5}, \frac{x - 2y}{5}\right)$$

Result: f is a function from A to B

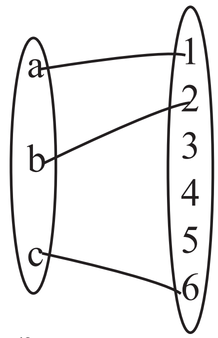
$|A| = n \qquad |B| = m$

Q. 1. If f is one-one

(a) $n \leq m$ (b) $n \geq m$ (c) $n = m$ (d) none n = 3 m = 3



one-one $\Rightarrow$ n = m



n m

n < m

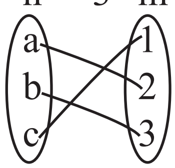
$\Big\rangle$ n $\leq$ m

Answer. (a)

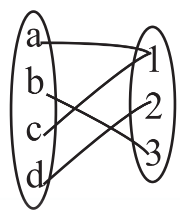
2. If f is onto then

(a) n $\leq$ m (b) n $\geq$ m (c) n = m (d) none

n = 3 m = 3



onto $\Rightarrow$ n = m



$\Big\rangle$ n $\geq$ m

onto n > m

n = 4 m = 3

Answer. (b)

3. If f is one-one and onto

(a) n $\leq$ m (b) n $\geq$ m (c) n = m (d) none

n $\leq$ m $\rightarrow$ one-one
n $\geq$ m $\rightarrow$ onto $\Big\rangle$ $\Rightarrow$ Both $\Rightarrow$ n = m

one-one and onto function are also called Bijecttion.

Q. Let G be a complete undirected graph on 6 vertices. If vertices of G are labeled, then the number of distinct cycles of length 4 in G is equal to

(a) 15 (b) 45 (c) 90 (d) 360

Ans. (b)

- From 6 vertices we can select 4 distinct vertices in $6c_4$ = 15 ways.

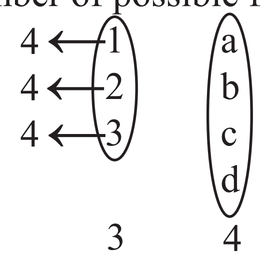
Now, with 4 vertices, we can form only 3 distinct cycles.

So, total number of distinct cycles of length 4 = 15 × 3 = 45

$\rightarrow$ Number of cyclic permutations of n objects = (n - 1)! and for n = 4, we get 31 = 6 ways.

But number of distinct cycles in a graph is exactly half the number of cyclic permutations as there is no left/right ordering in a graph. For example a - b - c - d and a - d - c - b are different permutations but in a graph they form the same cycle. So answer is 45.

IV. The number of possible function $|A| = n$, $|B| = m$, possible functions $= m^n$



Number of possible functions $= 4^3 = 64$

Ex: $|A| = 3$; $|B| = 5$

Number of functions $= 5^3 = 125$

GATE : 1996

Suppose X and Y are sets and |X| and |Y| are their respective cardinalities. It is given hat there are exactly 97 functions from X to Y, then

(a) $|X| = 1$, $|Y| = 97$ (b) $|X| = 97$, $|Y| = 1$ (c) $|X| = 97$, $|Y| = 97$

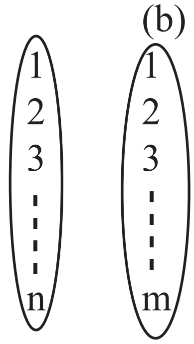
(d) None

Solution. (a)

$|Y|^{|X|} = (97)^1 = 97$

Q. Number of one-one function between $|A| = n$ $|B| = m$

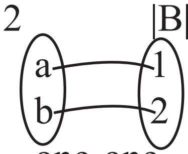
(a) n (b) m (c) ^mP_n (d) ^nP_m



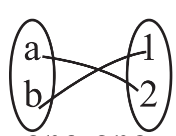
n permutations of m elements

$\boxed{^mP_n}$

$|A| = 2$ $|B| = 2$



one-one



one-one

$^2P_2 = 2$

* Number of onto functions $\sum_{i=0}^{m} {^mC_i} (-1)^i (m - i)^n$

Ex: $|A| = 3$ $|B| = 2$

$2c \ (2 - 0)^3 - 2c \ (2 - 1)^3 +$ $= 1 \times 2^3 - 2(1)^3 = 8 - 2 = 6$

Ex: $|A| = n$ $|B| = 2$

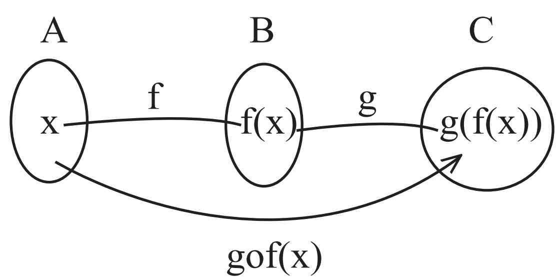
then number of onto functions $= 2^n - 2$

If is valid if B contain only two elements.

Composite Function:

$f : A \rightarrow B$ and $g : B \rightarrow C$

$gof : A \rightarrow C$ $gof(x) = g\, f(x)$



Q. $f : A \rightarrow B$ $g : B \rightarrow C$

$gof : A \rightarrow C$ defined

But fog not defined here.

Q. $f : R \rightarrow R$ $g : R \rightarrow R$

Both fog and gof are defined

But fog $\neq$ gof (need not be equal)

Ex: $f : R \rightarrow R$ $g : R \rightarrow R$

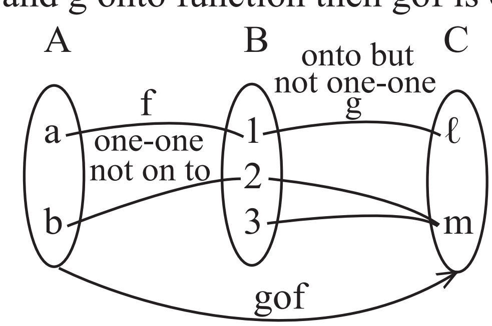
$f(x) = x^2$ $g(x) = x + 1$

$gof(x) = g(f(x)) = x^2 + 1$ $fog(x) = f(g(x)) = (x + 1)^2$

Important Result:

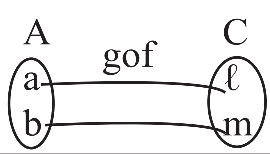
(1) If f and g one-one then the composition (gof) function is also one-one.

→ If f and g onto function then gof is onto.



$gof(a) = g(f(a)$ $\Rightarrow$ $g(1) = \ell$

$gof(b) = g(f(b))$ $\Rightarrow$ $g(2) = m$



gof is one-one and onto

2. → If gof is one-one then f will be one-one
→ If gof is onto then g will be onto
gof is one-one and onto then f is one-one and g is onto.

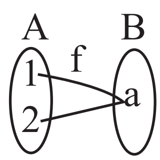
GATE : 2001

Let $f : A \rightarrow B$ be a function, and let E and F be subsets of A. Consider the following statements about images.

$s_1 : f(E \cup F) = f(E) \cup f(F)$ $\quad$ $s_2 : f(E \cap F) = f(E) \cap f(F)$

(a) only s_1 is correct
(b) only s_2 is correct
(c) both s_1 and s_2 are correct
(d) none of s_1 and s_2 is correct

Solution:



$A = \{1, 2\}$
$E = \{1\}$
$F = \{2\}$
$f(E) = a$ $\quad$ $f(F) = a$

$E \cup F = \{1, 2\}$ $\quad$ $E \cap F = \{\ \}$
$f(E \cup F) = \{a\}$ $\quad$ $f(E \cap F) = \{\ \}$
$f(E) \cup f(F)$ $\quad$ $f(E) \cap f(F)$
$a \cup a = \{a\}$ $\quad$ $a \cap a$
s_1 is correct. $\quad$ so s_2 is false.

GATE : 2015

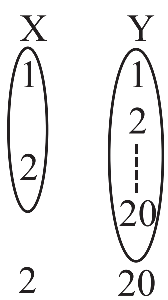
The number of onto functions (surjective functions) from set $x = \{1, 2, 3, 4\}$ to set $y = \{a, b, c\}$ is ________?

Ans. 36

GATE : 2015

Let X and Y denote the sets containing 2 and 20 distinct objects respectively and F denote the set of all possible functions defined from X and Y.

Let f be randomly chosen from F. The probability of F being one-one is ______?



Total number of functions $= 20^2 = 400$

Number of one-one function $= {}^{m}P_{n} = {}^{20}P_{2} = 380$

$\therefore$ Required probability $= \dfrac{380}{400} = 0.95$

Relations

Relation on set A

$R \leq A \times A$ $\quad$ cardinality of $|A \times A| = n^2$

A = {1, 2, 3, 4}   A = {1, 2, 3, 4}

A × A = {(1, 1) (2, 2) (3, 3) (4, 4) (1, 2) (1, 3) (1, 4) (2, 1) (2, 3) (2, 4) (3, 1) (3, 2) (3, 4) (4, 1) (4, 2) (4, 3)}

→ universal relation.

ϕ = { } → empty relation on A

'=' or Δ = {(1, 1) (2, 2) (3, 3) (4, 4)}

'<' = {(1, 2) (1, 3) (1, 4) (2, 3) (2, 4) (3, 4)}

'>' = {(2, 1) (3, 1) (3, 2) (4, 1) (4, 2) (4, 3)}

'≤' = {(1, 1) (1, 2) (1, 3) (1, 4) (2, 2) (2, 3) (2, 4) (3, 3) (3, 4) (4, 4)}

'≥' = {(1, 1) (2, 1) (2, 2) (3, 1) (3, 2) (3, 3) (4, 3) (4, 4)}

'\' = {(1, 1) (1, 2) (1, 3) (1, 4) (2, 2) (2, 4) (3, 3) (4, 4)}

↗ divides

A Relation R on A is

1. Reflexive

If (a, a) ε R   ∀a ε A

Yes → A × A, Δ, ≤, ≥, divides   No → ϕ, <, >

R_1 = {(1, 1) (2, 2) (3, 3)} Not reflexive

2. Irreflexive

A relation R on set A is called Irreflexive

If (a, a) and R   ∀ a ε A

Yes → ϕ, <, >

No → A × A, Δ, ≤, ≥, \

R	IR
✓	×

IR	R
✓	×

R = {(1, 1) (2, 2) (3, 3)}

Not Irreflexive

R	IR
×	?

3. Symmetric

A Relation R on set A is called symmetric

If (a, b) ε R then (b, a) ε R   where (a, b) ε A

Yes → A × A, ϕ, Δ   No → <, >, ≤, ≥, \

R_3 = {(1, 2) (2, 1) (1, 3)}   Not symmetric

Asymetric

If (a, b) ε R then (b, a) ∉ R   where a, b and A

Yes → ϕ, <, >   No → A × A, Δ, ≤, ≥, 1

R_2 = {(1, 2) (2, 1) (1, 3)}   Not Asymmetric

'ϕ' empty relation is both symmetric and asymmetric.

Antisymmetric

If (a, b) ε R and (b, a) ε R then a = b   where a, b ε A

Yes → ϕ, Δ, <, >, ≤, ≥, \   No → A × A

NOTE: I allow only Reflexive pairs but don't allow symmetric pair.

$R_1 = \{(1, 3)\}$ — Asymmetric, Antisymmetric

$R_2 = \{(1, 1)\ (1, 3)\}$ — Antisymmetric, Not Asymmetric

4. Transitive

If (a, b) ε R and (b, c) ε R then (a, c) ε R where a, b, c ε A

Yes → $A \times A$, ϕ, Δ, $<$, $>$, $\leq$, $\geq$, \

$R_1 = \{(1, 3)\ (3, 1)\}$ Not transitive

$R_1 = \{(1, 3)\ (3, 1)\ (1, 1)\}$ Not transitive

(1, 3) (3, 1) → (1, 1) | (3, 1) (1, 3) → (3, 3)

$R_1 = \{(1, 3)\ (3, 1)\ (1, 1)\ (3, 3)\}$

Let R, R_1, R_2 be Relations on A

	R, R_1, R_2	R^{-1}	$R_1 \cap R_2$	$R_1 \cup R_2$
1.	Reflexive	Reflexive	✓	✓
2.	Irreflexive	✓	✓	✓
3.	Symmetric	✓	✓	✓
4.	Antisymmetric	✓	✓	×
5.	Asymmetric	✓	✓	×
6.	Transitive	✓	✓	×

$A = \{1, 2, 3\}$

$R_1 = \{(1, 2)\}$ ↑ Asymmetric

$R_2 = \{(2, 1)\}$ ↑ Asymmetric

$R_1 \cup R_2 = \{(1, 2)\ (2, 1)\}$ Not Asymmetric

Ex:

$R_1 = \{(1, 2)\}$ Antisymmetric $R_2 = \{(2, 1)\}$ Antisymmetric

$R_1 \cup R_2 = \{(1, 2)\ (2, 1)\}$ Not Antisymmetric

NOTE:

1. Union of Asymmetric relation need not be Asymmetric.
2. Union of Antisymmetric relation need not be Antisymmetric.
3. Union of transitive relation need not be transitive.

Closure

1. Reflexive closure of R (R_r) :-

Smallest reflexive relation containing R

Ex: A = {1, 2, 3}   R = {(1, 1) (2, 2) (2, 3)}   Δ = {(1, 1) (2, 2) (3, 3)}

R_r = R ∪ Δ

= {(1, 1) (2, 2) (2, 3)}   {(1, 1) (2, 2) (3, 3)}

= {(1, 1) (2, 2) (3, 3) (2, 3)}

↑ Reflexive

2. Symmetric Closure (R_s)

$R_s = R \cup R^{-1}$

3. Transitive Closure:

A = {1, 2, 3, 4}   R = {(1, 2) (2, 1) (3, 4) (4, 3)}

Warshall's Algorithm

$$M_R^0 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 1 & 0 \end{array}$$

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & \textcircled{0} & 0 & 0 \\ 3 & 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 1 & 0 \end{array}$$

Modified as '1'

$$M_R^1 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & \textcircled{0} & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 1 & 0 \end{array}$$

Modified as '1'

$$M_R^2 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 1 & \textcircled{0} \end{array}$$

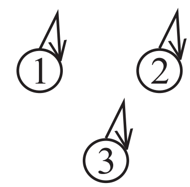
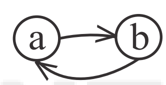
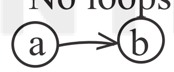
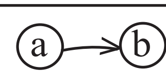
Modified as '1'

$$M_R^3 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 3 & 0 & 0 & \textcircled{0} & 1 \\ 4 & 0 & 0 & 1 & 1 \end{array}$$

Modified as '1'

$$M_R^4 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

$R^+ = \{(1, 1) (1, 2) (2, 1) (2, 2) (3, 3) (3, 4) (4, 3) (4, 4)\}$

M_R	Di-graph
Reflexive $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ All diagonal entries must be one.	 Loop at every vertex.
Irreflexive All diagonal entries must be zero.	'No loop'
Symmetric $M_R = M_R^T$	 loop allowed
Asymmetric $a_{ii} = 0$ and $a_{ij} = 1$ then $a_{ji} = 0$ $(i \neq j)$ $A = \{1, 2, 3\}$; $R = \{(1, 2) (1, 3)\}$ $M_R = \begin{bmatrix} 0 & \boxed{1} & 1 \\ \boxed{0} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	No loops  No edge from b to a.
Antisymmetric $a_{ij} = 1$ then $a_{ji} = 0$ $(i \neq j)$	 No edge from b to a

Equivalence Relation

A relation R on set A is said to be equivalence relation if the relation R is reflexive, symmetric and transitive.

Ex:

(1). $A = \{1, 2, 3, 4\}$ $R = \{(1, 1) (2, 2) (3, 3) (4, 4)\}$

1. Reflexive 2. Symmetric 3. Transitive

R is an Equivalence relation.

It is smallest equivalence relation.

(2). $A = \{1, 2, 3, 4\}$

$\{(1, 1) (1, 2) (2, 1) (2, 2) (3, 3) (3, 4) (4, 3) (, 4)\}$

1. Reflexive 2. Symmetric 3. Transitive

Equivalence Class:-

Let R be an equivalence relation on A and let a ε A. The equivalence class of a, denoted by (a) or $\bar{a}$ is defined as

$$[a] = \{b \in A \mid (a, b) \in R\}$$

Ex:

$A = \{1, 2, 3, 4\}$ $R_1 = \{(1, 1) (2, 2) (3, 3) (4, 4)\}$

$[1] = \{1\}$; $[2] = \{2\}$; $[3] = \{3\}$; $[4] = \{4\}$

$R_2 = \{(1, 1) (1, 2) (2, 1) (2, 2) (3, 3) (3, 4) (4, 3) (4, 4)$

$[1] = \{1, 2\}$; $[2] = \{1, 2\}$; $[3] = \{3, 4\}$; $[4] = \{3, 4\}$

Properties

(1) a ε [a] because of reflexive property.

(2) b ε [a] then a ε [b]

(3) b ε [a] then [a] = [b]

(4) $[a] = [b]$ or $[a] \wedge [b] = \phi$

Partition of a set:

$A \neq \phi$ $P \neq \phi = \{A_1, A_2, A_3, \ldots\ldots\ldots\ldots A_n\}$ $A_i \neq \phi$

is called partition of A if

(1) $A = A_1 \cup A_2 \cup A_3 \ldots\ldots\ldots\ldots A_n$

(2) $A_i \cap A_j = \phi \quad (i \neq j)$

Ex: $A = \{1, 2, 3, 4\}$ $P = \{(1, 2) (3, 4)\}$

(1) $\{1, 2\} \cup \{3, 4\} \Rightarrow \{1, 2, 3, 4\}$

(2) $\{1, 2\} \cap \{3, 4\} \Rightarrow \phi$

2-Part Partition

Q. Which of the following is not a valid partition?

(a) $\{\{1\}, \{2\}, \{3\}\}$ (b) $\{\{1\}, \{2\}, \{3\}, \{4\}\}$

(c) $\{\{1, 2, 3, 4\}\}$ (d) $\{\{1, 2\}, \{3\}, \{3, 4\}\}$

(a) → 3 part partition (b) → 4 part partition (c) → 1 part partition

(d) Not a valid partition

Ex: $A = \{1, 2, 3, 4\}$ $R = \{(1, 1)(2, 2)(3, 3)(4, 4)\}$

$[1] = \{1\}$; $[2] = \{2\}$; $[3] = \{3\}$; $[4] = \{4\}$

$P = \{\{1\}, \{2\}, \{3\}, \{4\}\}$

Ex: $A = \{1, 2, 3, 4\}$

$\{(1, 1) (1, 2) (2, 1) (2, 2) (3, 3) (3, 4) (4, 3) (4, 4)\}$

$[1] = \{1, 2\} = [2]$ $[3] = \{3, 4\} = [4]$

$P = \{\{1, 2\}, \{3, 4\}\}$ $\rightarrow$ 2 part partition

Given an equivalence relation 'R' on set A, we can find a unique partition on A
$\rightarrow$ the part of partition are distinct equivalence classes ———— (1)

Ex: $A = \{1, 2, 3, 4\}$ $P = \{\{1\}, \{2\}, \{3\}, \{4\}\}$

$\{1\} \rightarrow \{(1, 1)\}$ $\{2\} \rightarrow \{(2, 2)\}$ $\{3\} \rightarrow \{(3, 3)\}$

$\{4\} \rightarrow \{(4, 4)\}$

Equivalence relation: $\{(1, 1) (2, 2) (3, 3) (4, 4)\}$

Ex: $A = \{1, 2, 3, 4\}$ $P = \{\{1, 3\}, \{2, 4\}\}$

$\{1, 3\} \rightarrow \{(1, 1) (1, 3) (3, 1) (3, 3)\}$

$\{2, 4\} \rightarrow (2, 2) (2, 4) (4, 2) (4, 2)$

$E{\cdot}R = \{(1, 1) (1, 3) (3, 1) (3, 3) (2, 2) (2, 4) (4, 4) (4, 2)\}$

Given a partition P on set A we can find a unique equivalence relation on A — (2)
from (1) & (2)
The exist a one to one correspondance between number of partition on A and number of equivalence relation on A.
If $|A| = n$ then
Number of equivalence relation on A = Number of partitions on A = Bell number (B_n)

$A = \{1\}$ $A = \{1, 2\}$

$E{\cdot}R = \{(1, 1)\}$ $E{\cdot}R = \{(1, 1) (2, 2)\}$

$[1] = \{1\}$ $E{\cdot}R = \{(1, 1) (1, 2) (2, 1) (2, 2)$

↑ Partitions = $\underbrace{\{\{1\}, \{2\}\}}_{\text{2 Partition}}$

1- Partition

Let $|A| = n$

1. Total number of relations $= 2^{n^2}$
2. Total number of reflexive relations $= 2^{n(n - 1)}$
3. Total number of irreflexive relations $= 2^{n(n - 1)}$
4. Total number of symmetric relations $= 2^{\frac{n(n + 1)}{2}}$
5. Total number of asymmetric relations $= 3^{\frac{n(n - 1)}{2}}$

6. Total number of antisymmetric relations = $2^{n} \cdot 3^{\frac{n(n-1)}{2}}$
7. Transitive = No closed formula

1. Total number of relations:- A = {1, 2}

$A \times A = \{(1, 1)\ (1, 2)\ (2, 1)\ (2, 2)\}$

(1, 1)	(1, 2)	(2, 1)	(2, 2)
↓	↓	↓	↓
2	2	2	2

$\Rightarrow$ $2^4 = 16$ Relations

Selected / Not selected ⟩ 2

Number of relations: - $|A| = n$ $|A \times A| = n^2$

1	2	3		n				n^2
(1, 1)	(2, 2)	(3, 3)	- - -	(n, n)	(1, 2)	(2, 1)	- - -	-
↓	↓	↓		↓	↓	↓		↓
2	2	2		2	2	2		2

n^2 times

Number of relations = 2^{n^2} = number of subsets of $A \times A$

2. Number of reflexive relations:-

(1, 1)	(2, 2)	- - - (n, n)	(1, 2)		-
↓	↓	↓	↓		
1	1	1	2		

these must be present (for the 1's); $n^2 - n$ (for the 2's)

↓

$$(1)^n \times 2^{n^2 - n} = 2^{n^2 - n} = 2^{n(n-1)}$$

A = {1, 2}

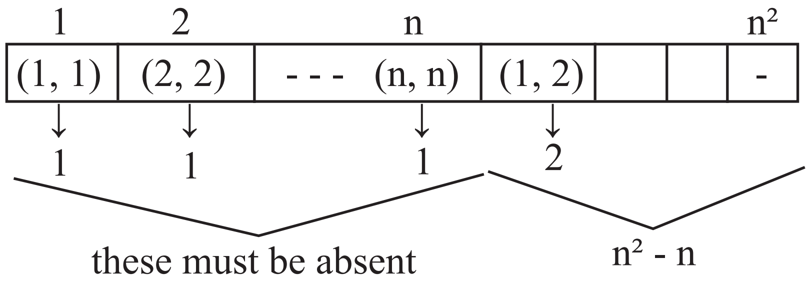
Number of reflexive relations = {(1, 1) (2, 2)}

= {(1, 1) (2, 2) (1, 2)} = {(1, 1) (2, 2) (2, 1)}

= {(1, 1) (2, 2) (1, 2) (2, 1)

$2^{2(2-1)} = 4$

3. Number of Irreflexible relations



$$(1)^n \times 2^{n^2 - n} = 2^{n(n - 1)}$$

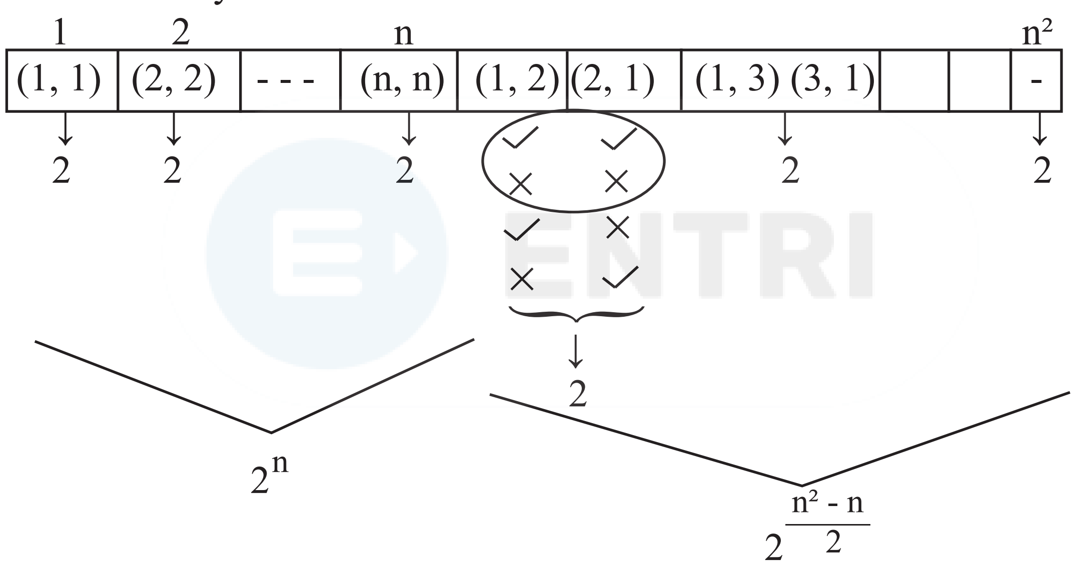
A = {1, 2}

$R_1 = \{(1, 2)\}$ $\quad R_2 = \{(2, 1)\}$ $\quad R_3 = \{(1, 2) (2, 1)\}$

$R_4 = \phi$

$$2^{2(2 - 1)} = 4$$

4. Number of symmetric relations



$$2^{n + \frac{n^2 - n}{2}} = 2^{\frac{2n + n^2 - n}{2}} = 2^{\frac{n^2 + n}{2}} = 2^{\frac{n(n + 1)}{2}}$$

A = {1, 2}

Number of symmetric relations =

$R_1 = \phi$; $\quad R_2 = \{(1, 1)\}$; $\quad R_3 = \{(2, 2)\}$; $\quad R_4 = \{(1, 2) (2, 1) (1, 1)\}$

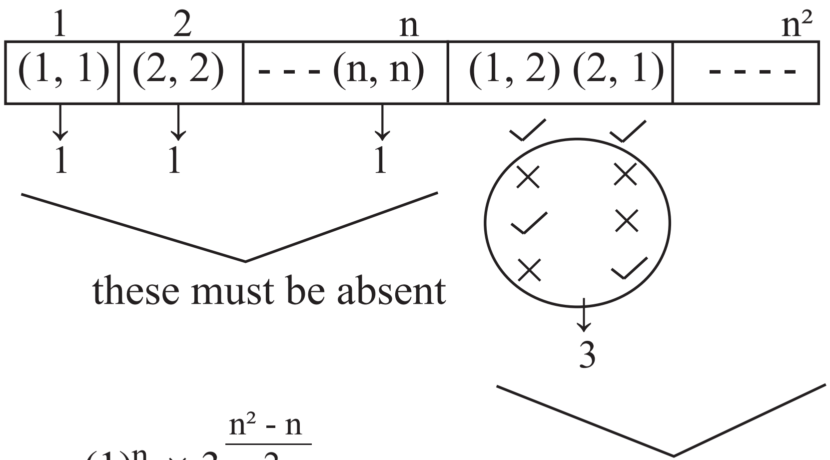
$R_5 = \{(1, 2) (2, 1) (2, 2)\}$; $\quad R_6 = \{(1, 2) (2, 1) (1, 1) (2, 2)\}$;

$R_7 = \{(1, 2) (2, 1)\}$; $\quad R_8 = \{(1, 1) (2, 2)\}$

Total = 8

$$2^{\frac{n(n + 1)}{2}} = 2^{\frac{2 \times 3}{2}} = 2^3 = 8$$

5. Number of asymmetric relation : -



$$\frac{n^2 - n}{2}$$

$$(1)^n \times 3^{\frac{n^2 - n}{2}}$$

$$= 3^{\frac{n^2 - n}{2}}$$

$A = \{1, 2\}$

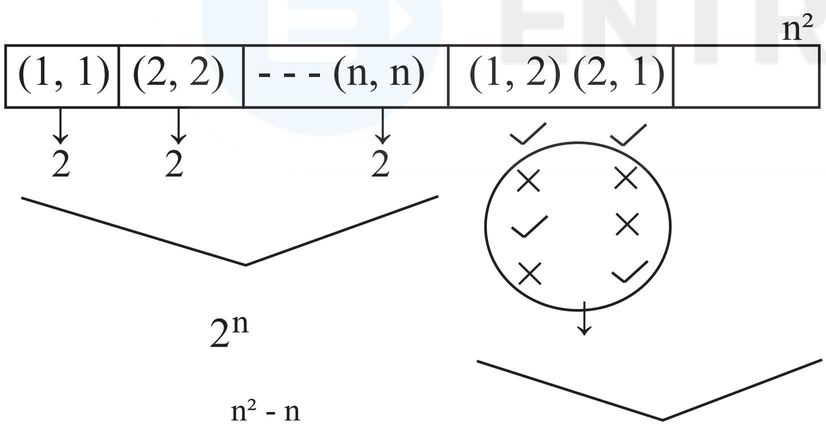
$R_1 = \{(1, 2)\}$

$R_2 = \{(2, 1)\}$

$R_3 = \phi$

$$3^{\frac{4 - 2}{2}} = 3' = 3$$

6. Number of antisymmetric relation



$$3^{\frac{n^2 - n}{2}}$$

$$= 2^n \times 3^{\frac{n^2 - n}{2}}$$

Q. Consider a set containing n elements then how many relations are symmetric as well as reflexive.

Solution:

Number of symmetric relations $\quad 2^n \times 2^{\frac{n(n - 1)}{2}}$

Number of symmetric and reflexive

$$(1)^n \times 2^{\frac{n(n - 1)}{2}} = 2^{\frac{n(n - 1)}{2}}$$

ENTRI

Bell Number:

$|A| = n$

Number of partitions on A = Number of equivalence relations on A = B_n

$$B_n = \sum_{r=1}^{n} s(n, r)$$

s(n, r) is defined as $s(n, 1) = s(n, n) = 1$

$s(n, r) = s(n - 1, r - 1) + rs(n - 1, r)$

$$B_1 = \sum_{r=1}^{1} s(1, r) = s(1, 1) = 1$$

$$B_2 = \sum_{r=1}^{2} s(2, r) = s(2, 1) + s(2, 2) = 1 + 1 = 2$$

$$B_3 = \sum_{r=1}^{3} s(3, r) = s(3, 1) + s(3, 2) + s(3, 3) = 1 + 3 + 1 = 5$$

$s(3, 2) = s(3 - 1, 2 - 1) + 2s(3 - 1, 2) = s(2, 1) + 2s(2, 2) = 1 + 2 \times 1$
$= 3$

$B_4 = 15$

Ex: A = {1, 2, 3}

Number of partition on A = B_3

(a) 1 (b) 2 (c) 5 (d) 15

Answer. (c)

Partial Ordered Relation (POR)

(1) Reflexive (2) Antisymmetric (3) Transitive

Q1. '≤' Relation on 2 is

(a) Reflexive (b) Antisymmetric (c) Transitive

(d) Partial order relation

Reflexive: $a \le a$

Antisymmetric: $a \le b \;\&\&\; b \le a \Rightarrow a = b$

Transitive: $a \le b \;\&\&\; b \le c \Rightarrow a \le c$

ENTRI

Partial Order Relation

Ex: '\' Relation on 2 is

(a) only reflexive (b) only antisymmetric (c) only transitive

(d) POR

Reflexive

$\frac{o}{o}$ not defined so not reflexive.

Antisymmetric

$\frac{-1}{1} = -1$ $\frac{1}{-1} = -1$

but $1 \neq -1$ not antisymmetric

Transitive

$\frac{a}{b}$ & $\frac{b}{c}$ $\Rightarrow$ $\frac{a}{c}$ Integer transitive

So, option (c) is true.

Q. '1' relation on 2^{+} is (→ set of positive integer)

(a) only reflexive (b) only antisymmetric (c) only transitive

(d) POR

Yes, it is POR.

Q. '$\subseteq$' on P(s)

(a) only reflexive (b) only antisymmetric (c) only transitive

(d) POR

` Reflexive: Every set is subset of itself.

$A \subseteq A$

Antisymmetric: $A \subseteq B$ but $B \nsubseteq A$ so it is antisymmetric.

Transitive: $A \subseteq B$ && $B \subseteq C$ so $A \subseteq C$. It is transitive. So it is POR.

A is set and R is relation on A

R	R^{-1}
Reflexive	Reflexive
Antisymmetric	Antisymmetric
Transitive	Transitive
POR	POR

Comparable:-

Let $<A, \underline{\alpha}>$ be a poset

Two element a, b of set A are said to be comparable if either

$a \ \underline{\alpha} \ b$ or $b \ \underline{\alpha} \ a$

↑ related to or comparable ↑ related to or comparable

Ex: $A = \{1, 2, 3, 4\}$, '$\leq$'

2, 3 are comparable $2 \leq 3$

(4, 1) are comparable $4 \nleq 1$ but $1 \leq 4$

Every pair of element is comparable here.

Toset.

Ex: $A = \{1, 2, 3, 4\}$, '\'

1/2 ⟶ comparable; 2/4 ⟶ comparable;

2, 3 ⟶ comparable × because $\frac{2}{3}$ × $\frac{3}{2}$ ×

Poset.

Total Ordered Set (TOSET):-

A poset $<P, \underline{\alpha}>$ in which every pair of element are comparable is called Toset.

$A = \{1, 2, 3, 4, 5\}$, '$\leq$'

$\underbrace{1 \leq 2 \leq 3 \leq 4 \leq 5}_{\text{totally comparable}}$

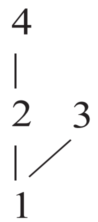
Hasse Diagram:-

Ex: $A = \{1, 2, 3, 4\}$, '$\leq$'

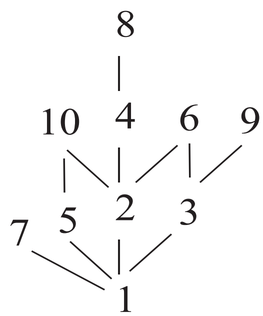
4
|
3
|
2
|
1

Hasse diagram of Toset is like a chain.

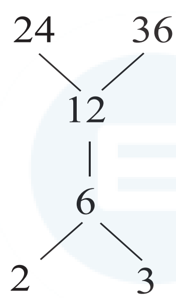
Ex: A = {1, 2, 3, 4}, ‘\’



A = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}, ‘\’ → divide relation.



Ex: A = {2, 3, 6, 12, 24, 36}, ‘\’ (divide)
Number of lines in H.D.



Number of lines = 5

D_V = set of all positive divisor, m, n and 2^+ → <D_V, 1> is a poset

(a) a/a Reflexive

(b) a/b and b/a then a = b (Antisymmetric)

(c) a/b, b/c then a/c (Transitive)

<D_8, 1> construct Hasse diagram.

D_8 = {1, 2, 4, 8}

Ex: <T, $\underline{\alpha}$ >, T is having 6 elements. |T| = 6, number of lines in H.D.

(a) 3 (b) 4 (c) 5 (d) 6

NOTE: Hasse diagram of a toset is like a chain.

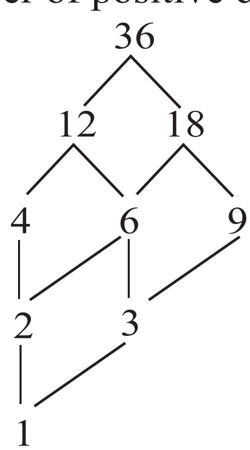
Number of elements = n
then number of lines = n - 1

Q. The Hasse diagram for following set with division <P, 1> poset.
A = {2, 3, 5, 7}, '/' (divides)
and number of lines = ?

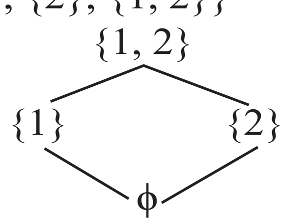
Solution:

Number of lines = 0

Ex: $<D_{36}, 1>$
Number of positive division of 36 = {1, 2, 3, 4, 6, 9, 12, 18, 36}

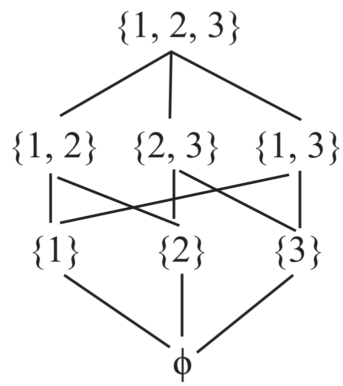


Ex: A = {1, 2} $< P(s), \subseteq >$ poset
P(s) = {ϕ, {1}, {2}, {1, 2}}



Ex: $A = \{1, 2, 3\}$ $<P(A), \subseteq>$

try at home



Special Elements

Maximal and minimal elements

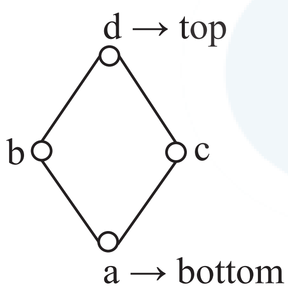
An element of a poset is called minimal if it is not greater than any element of the poset.

i.e., a is minimal if there is no element b ε s such that b α a

↑ related to

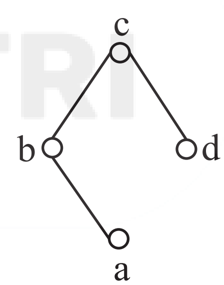
By default maximal and minimal elements are top and bottom elements respectively in the Hasse diagram.

(1)



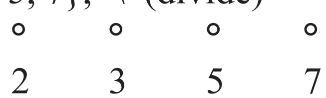
maximal = d
minimal = a

(2)



maximal = c
minimal = a, d

(3) {2, 3, 5, 7}, '\' (divide)



Maximal elements = 4; Minimal elements = 4

→ Every poset has a maximal and minimal elements → false

Ex: <2,   > is a poset

Maximal → No maximal; Minimal → No minimal

→ Every finite poset has atleast one maximal and minimal element → True

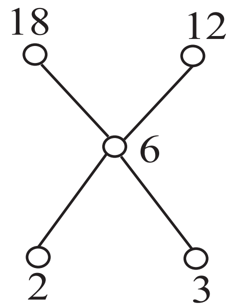
→ Maximal and minimal elements, if exist, are unique → false

Greatest and Least Element:-

→ If maximal is unique (only one) then that is greatest.

→ If minimal is unique (only one) then that is least.

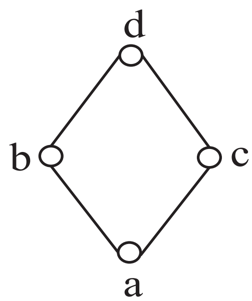
(1)



Maximal → 12, 18   Greatest → none

Minimal → 2, 3   Least → none

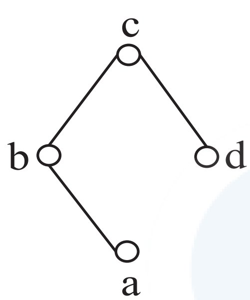
(2)



Greatest = d

Least = a

(3)

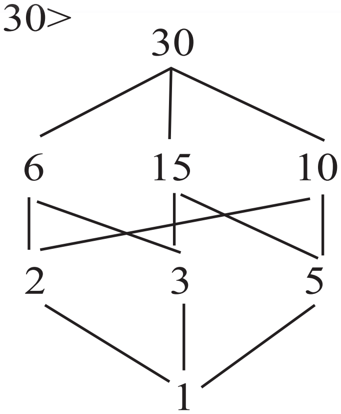


Greatest = c

Least = none → minimal = a, d

(4) <D_{30} , 1>

<1, 2, 3, 5, 6, 10, 15, 30>



Least = 1

Greatest = 30

NOTE: $|D_n|$

Least = 1;   Greatest = n

NOTE: The greatest and least element, if exist then unique → true.

Upper bound and Lower bound

Ex: A = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}, '≤'

<A, $\underline{\alpha}$ > is a poset.   P = {3, 4, 5} ⊆ A

least upper bound
↑
Upper bound of P = ⑤, 6,7, 8, 9, 10
Lower bound of P = 1, 2,③
↓
greatest lower bound

Ex: <P, $\underline{\alpha}$ > poset
{a, b} $\subseteq$ P → two element subset

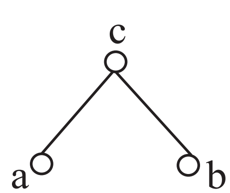
L u b {a, b}	$a \cup b$	a join b
g ℓ b {a, b}	$a \cap b$	a meet b

Ex: <{1, 2, 3, 4, 5}, ≤> poset
L u b {1, 2} = 2 = $1 \cup 2$
g ℓ b {1, 2} = 1 = 1   2

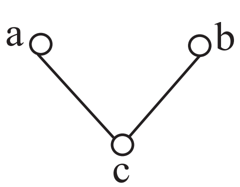
L u b ↙
$3 \cup 4 = 4$ $1 \cup 4 = 4$
$3 \cap 4 = 3$ $1 \cap 4 = 1$
↑
g ℓ b

4 - L u B
|
3
|
2
|
1 - g ℓ B

Lattice:-
A poset <P, $\underline{\alpha}$ > in which every pair of element {a, b} has a L u b & g ℓ b is called Lattice.
Finding L u b & g ℓ b of the non-comparable elements:-

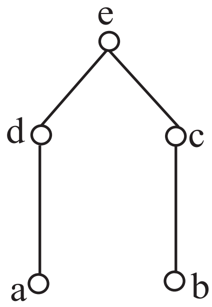
(1)


L u b (a, b) = a join b = $a \cup b$ = c

(2)


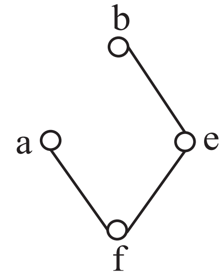
g ℓ b (a, b) = a meet b = $a \cap b$ = d

(3)



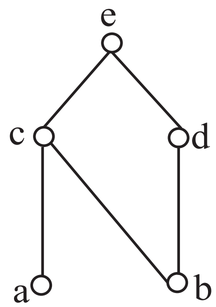
$a \cup b = e$

(4)



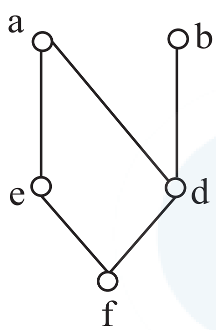
$a \cap b = f$
↑
meet

(5)



$e \rightarrow u^b$
$a \cup b = c$
↑
L u b

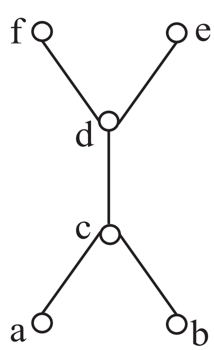
(6)



$a \cap b = d$ - g ℓ b
f - ℓ b (lower bound)

Q. Which of the following an lattice.

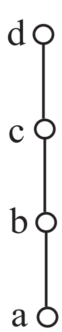
(1)



$e \cup f$ = does not exist; $a \cap b$ = does not exist

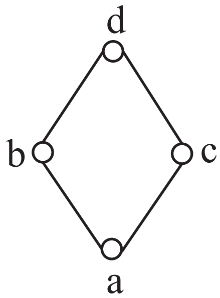
NOTE: No open structure can be a lattice. → not a lattice.

(2)



NOTE: Every toset is a lattice.

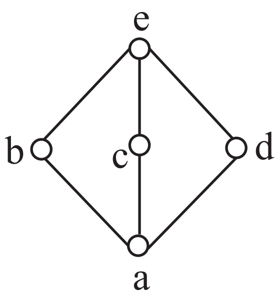
3.



Lattice

$b \cup c = d$
$b \cap c = a$

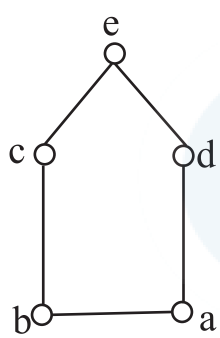
(4)



Diamond lattice

$b \cup c = e$; $b \cap c = a$; $c \cup d = e$;
$c \cap d = a$; $b \cup d = e$; $b \cap d = a$

(5)

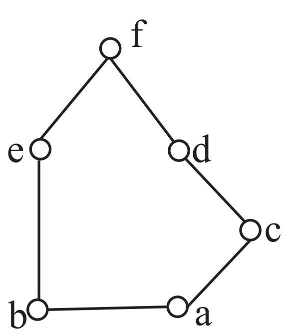


$c \cup d = e$
$c \cap d = b, a$

If more than one L u b & g ℓ b exist then they must be comparable for being a lattice.
So it is a lattice.

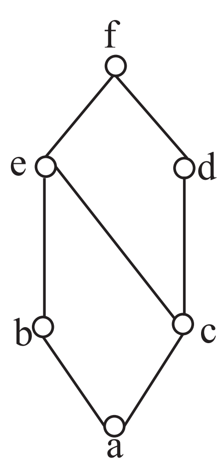
If an edge is exist between two different elements then they must be comparable.

(6)



lattice

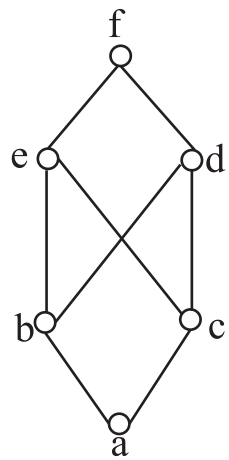
(7)



lattice

$b \cup c = e$; $b \cap c = a$;
$d \cup e = f$; $d \cap e = c$;
$b \cup d = e$; $b \cap d = a$

(8)

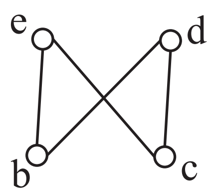


$b \cup c = \underbrace{d, e}$
Not comparable

$d \cap e = \underbrace{b, c}$
Not comparable

So not a latice.

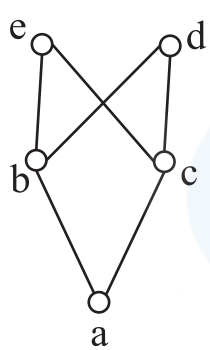
(9)



not a lattice

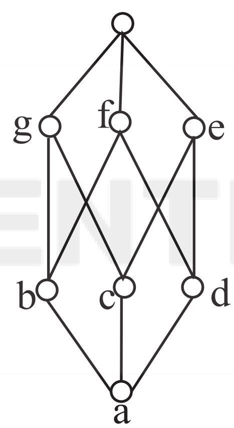
$d \cup e$ = does not exist.

(10)



not a lattice

(11)



lattice

$f \cup g = h$;
$f \cap g = b$;
$c \cup d = e$;
$c \cap d = a$

Let <L, $\underline{\alpha}$ > be a lattice then the following properties hold.

(1) Idempotent

$a \cup a = a$ $\quad$ $a \cap a = a$

(2) Commutative

$a \cup b = b \cup a$ $\quad$ $a \cap b = b \cap a$

(3) Associative

$a \cup (a \cap b) = a$ $\quad$ $a \cap (a \cup b) = a$

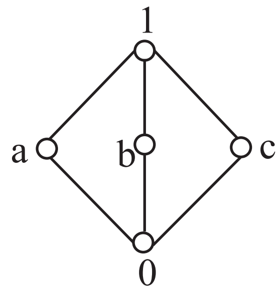
Q. Which of the following properties are not satisfied by lattice?

(a) Commutative (b) Distributive (c) Absorption

(d) Idempotent

Answer. (b)

Ex:



Diamond lattice

$a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$

$a \cup 0 = 1 \cap 1$

$a \neq 1$

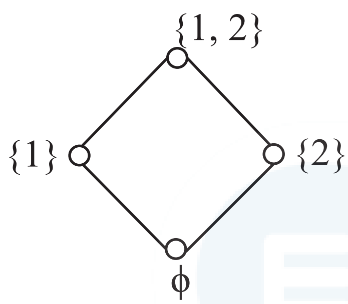
not holds

In any lattice the following distributing inequalities holds:-

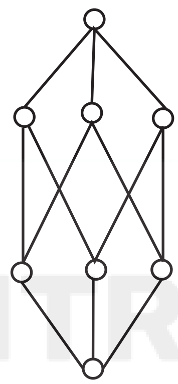
(A) $a \cup (b \cap c) \ \underline{\alpha}\ (a \cup b) \cap (a \cup c)$ (B) $(a \cap b) \cup (a \cap c) \ \underline{\alpha}\ a \cap (b \cup c)$

→ A lattice $<L, \underline{\alpha}>$ in which distributive properties are satisfied is called Distributive lattice.

Ex: Distributive lattice



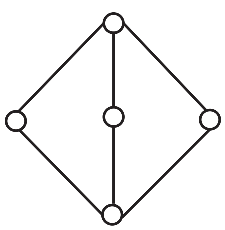
(2)



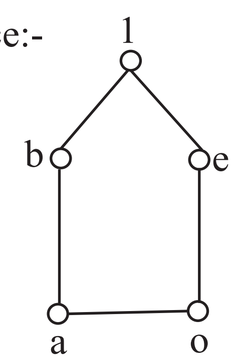
Sublattice:-

$<L, \underline{\alpha}>$ is a lattice; $S \subseteq L$; $<S, \underline{\alpha}>$ is a lattice; then it is called Sublattice.

Two important non-distributive lattice:-



Diamond



Pentagon

Result: A lattice $<L, \underline{\alpha}>$ is non-distributive if it contains sub-lattice isomorphic to diamond or pentagon.

Bounded Lattice

A lattice $<L, \underline{\alpha}>$ in which greatest and least element exist.

→ Every finite lattice is bounded.

Ex: $<2, \subseteq>$ is a lattice. But not bounded.

Greatest = 1; Least = 0

Complement of a element : -

Let $<L, \underline{\alpha}>$ be bounded lattice

An element a ε L is complement of b ε L.

if $a \cup b = 1 \rightarrow$ greatest; $a \cap b = 0 \rightarrow$ least

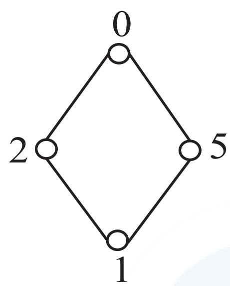
(1) If a is complement of b then b is complement of a.

(2) ‘1’ & ‘0’ an complement of each other.

$0 \cup 1 = 1; \quad 0 \cap 1 = 0$

Ex: Find all complements.

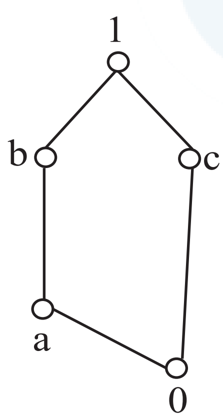
(1)



Element	Complement
1	10
10	1
2	5
5	2

$2 \cup 5 = 10 =$ greatest; $2 \cap 5 = 1 =$ least

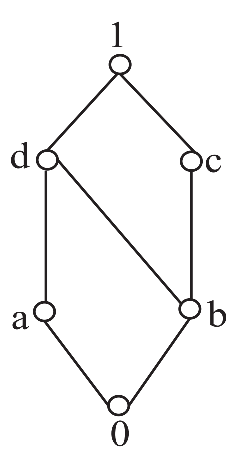
(2)



Element	Complement
0	1
1	0
a	c
b	c
c	a, b

$a \cup c = 1; \quad a \cap c = 0$

(3)



Element	Complement
0	1
1	0
a	c
c	a

$b \cup d = d \rightarrow$ not greatest

$b \cap d = b \rightarrow$ not least

b, d don’t have complement.

→ Complement may or may not exist.

→ Complement, if exist, are not necessary unique.

Result:

→ In a distributive lattice, complement if exist, are unique.

→ A bounded lattice in which complement of every element exist is called complemented lattice.

→ Bounded, distributive and complemented lattice is Boolean algebra.