CHAPTER 3

THEORY OF MACHINES

Machine is a mechanism that consists of fixed and moving parts and modifies mechanical energy to assist in performing of human tasks. *Theory of machines* is an applied science of the relationships between geometry and relative motion of the parts of machine, and concerns to the forces which act on those parts. It involves analysis of as well as synthesis. Analysis is the study of motions and forces concerning different parts of an existing mechanism, whereas synthesis involves design of different parts. The study of mechanisms, therefore, is divided into kinematics and dynamics. The branch of *kinematics* deals with the relative motions of different parts of a mechanism without taking into account the forces producing the motions, thus it is solely based on geometric point of view. *Dynamics* involves determination of forces impressed upon different parts of a mechanism. It has sub-branches of kinetics and statics. *Kinetics* is study of forces when the body is in motion whereas *statics* deals with forces when the body is stationary.

3.1 MECHANISMS AND MACHINES

Mechanisms and machines are composed of kinematic links and pairs. Mechanisms modify the external force and deliver some advantage in motion and force. For this, a term *mechanical advantage* is defined for a mechanism as the ratio of the output force (or torque) to the input force (or torque). The power input and output remain the same during such modification.

3.1.1 Rigid and Resistant Bodies

A *rigid body* does not suffer any distortion under the action of external forces. *Resistant bodies* are semi-rigid bodies, normally flexible but under certain loading conditions, act as rigid bodies. Resistant bodies constitute parts of machines through which requisite motion and forces are transmitted, such as belts, fluids, springs. For example, a belt is rigid when subjected to tensile force. Similarly, fluid acts as rigid body in hydraulic press during compression. Same is the case with springs.

3.1.2 Kinematic Links

Kinematic link is a resistant body or an assembly of resistant bodies which go on to make a part of a machine and enable modification and transmission of mechanical work through the relative motion between the parts. Each link or element can consist of several parts which are manufactured as separate units. A link need not necessarily be a rigid body, but it must be a resistant body.

Depending upon the number of joints on which turning pairs can be placed, kinematic links are classified as *binary*, *ternary*, *quaternary* links, which have 2, 3, 4 joints, respectively [Fig. 3.1].

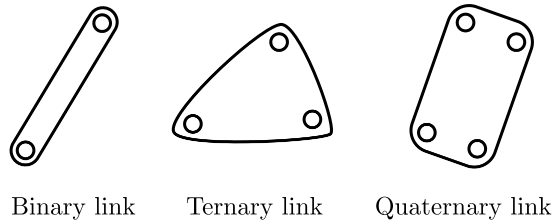


Figure 3.1 | Types of links.

A *structure* only modifies and transmits forces without resulting in any work. The simplest structure is one with three bars.

3.1.3 Kinematic Pairs

Two links when connected together in such a way that their relative motion is completely or successfully constrained, constitute a *kinematic pair*.

3.1.3.1 Types of Constrained Motions A kinematic pair can be constituted by various means which decide the type of relative motion between the links. If this relative motion is one and only type, then it is said to be *constrained motion*. This can be of three types:

1. *Completely Constrained Motion* A *completely constrained motion* takes place in one definite direction. The motion is complete by its own links. For example, a square bar can only slide in a square slot.

2. *Incompletely Constrained Motion* If motion of a link is possible in more than one direction and governed by the direction of force, the motion is called *incompletely constrained motion*. For example, a circular bar can rotate and reciprocate in a round hole.

3. *Successfully Constrained Motion* When incompletely constrained motion is made to be only one direction by using some external means, it is called *successfully constrained motion*. For example, the vertical motion of a shaft in footstep bearing is constrained by load upon it while it can undergo rotation only. Similarly, the rotatory motion of a piston inside the cylinder is constrained by a piston pin.

3.1.3.2 Classification of Kinematic Pairs Kinematic pairs are classified according to the nature of relative motion, contact, and constraint. This is explained as follows:

1. *Nature of Relative Motion* Various types of kinematic pairs are explained as follows [Fig. 3.2]:

   (a) *Sliding Pair* - If two links of a pair are connected in such a way that they can have only sliding motion, they are called *sliding pair*, such as piston-cylinder, ram-guide [Fig. 3.2]. A sliding pair has a completely constrained motion.

   (b) *Turning Pair* - If two links of a pair are connected in such a way that they can have only turning motion, they are called *turning pair*, such as a shaft with collars at both ends fitted into a circular hole, and the crankshaft in a journal bearing. A turning pair has a completely constrained motion.

   (c) *Rolling Pair* - When a link of a pair has a rolling motion relative to other, the pair is called *rolling pair*, or *cylindrical pair*, for example, a rolling wheel on a flat surface, pulley in a belt drive. The ball-bearing shaft constitute a very interesting example in which balls of bearing make rolling pair with both the shaft and the bearing.

   (d) *Screw Pair* - If a link of a pair has a turning as well as sliding motion relative to the other, the pair is called *screw pair*. The lead screw and nut of a lathe machine constitute screw pair.

   (e) *Spherical Pair* - When one link with spherical interface turn inside another link, it forms *spherical pair*, for example, ball and socket joint.

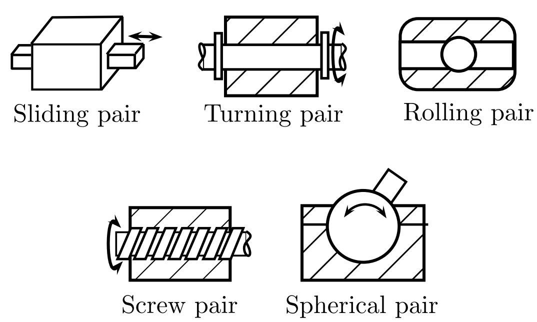


Figure 3.2 | Relative motion in kinematic pairs.

2. *Nature of Contact* By virtue of the nature of contact, kinematic pairs can be two types:

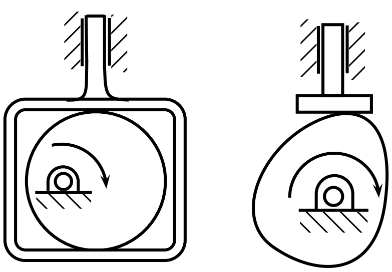
   (a) *Lower Pair* - If the two links in a pair have surface contact while in motion, the pair so formed is called a *lower pair*. The relative motion is purely sliding or turning, and the contact surface is similar in both links, for example, shaft revolving in a bearing, steering gear mechanism, universal coupling.

(b) *Higher Pair* - If the two links in a pair have point or line contact while in motion, the pair so formed is known as a *higher pair*. The contact surface of the two links is dissimilar. For example, cam and follower mechanism, toothed gears, ball and roller bearings.

3. Nature of Constraint The concept of closed and unclosed kinematic pairs is explained as follows [Fig. 3.3]:

(a) *Closed Pair* - If the elements of a pair are held together mechanically, it is known as a *closed pair*. The contact between the two elements can be broken only by destruction of at least one of them. All lower pairs and some of the higher pairs are closed pairs.

(b) *Unclosed Pair* - If the elements of a pair are not held together mechanically, instead, either due to force of gravity or some spring action, it is called *unclosed pair*. For example, cam and follower.



Closed pair Unclosed pair

Figure 3.3 Nature of constraints.

3.1.4 Kinematic Chains

A *kinematic chain* is a combination of kinematic pairs in which each link constitutes two pairs and its relative motion is completely constrained. Thus, a kinematic chain has a single degree of freedom [*Section* 3.1.9]. A *redundant chain* does not allow any motion of a link relative to the other.

A simplest kinematic chain is a four bar chain. A chain having more than four links is called *compound kinematic chain*.

3.1.5 Inversions of Kinematic Chain

Primary function of a mechanism is to transmit or to modify motion and it can work as a machine. Different types of motions are possible from a given mechanism by fixing one of its kinematic links. The mechanisms obtained in this way can be very different in appearance and in the purposes for which they are used. Each mechanism is termed as the *inversion* of the original kinematic chain.

As many inversions are possible as the number of links in the mechanism. Inversion has no effect on relative motion between links of the mechanism, but changes the absolute motion.

3.1.6 Four-Bar Chains

A *four-bar mechanism* consists of four bars joined together in closed series with pin joints [Fig. 3.4]. The four links can have different lengths solely depending on the purpose of inversions. Let l_1, l_2, l_3, l_4 be the lengths

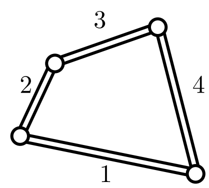


Figure 3.4 Four-bar chain mechanism.

of links of the four-bar mechanism in ascending order ($l_1 < l_2 < l_3 < l_4$). Based on the lengths of the fixed and free links, the inversions of this four-bar mechanism can be divided into two groups:

1. Class I According to *Grashof's law*, if there is to be continuous relative motion or rotation between two links, the sum of the shortest and largest links of a planar four-bar linkage cannot be greater than the sum of remaining two links.

$$l_1 + l_4 \leq l_2 + l_3$$

This type of mechanism is called of class-I, which comprises the following mechanisms:

(a) *Crank Lever Mechanism* - By making the largest link as crank (adjacent link fixed).

(b) *Drag Link Quick Return Mechanism* - By fixing the shortest link.

(c) *Double Crank Mechanism* - By having opposite links of equal length and fixing the shortest link.

2. Class II This inversion is opposite to Class-I in reference to Grashof's law.

$$l_1 + l_4 > l_2 + l_3$$

Therefore, there is no continuous rotation between the two links, and the resulting mechanism is a *rocker-rocker mechanism* or *double rocker mechanism*.

Therefore, in a four-bar chain, for the link adjacent to the (fixed) short link to be a crank, the sum of the shortest and the longest links should be less than the sum of the other two links.

Various inversions of a four-bar chain are described as follows:

1. *Crank-Lever Mechanism* A *crank-lever mechanism*, also known as *crank-rocker mechanism* is obtained by fixing the largest link of the four-bar chain.

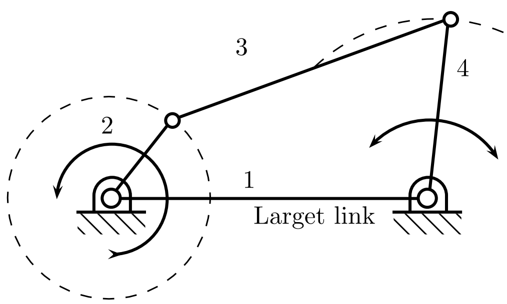


Figure 3.5 | Crank-lever mechanism.

The crank rotates full revolution but the follower can only oscillate [Fig. 3.5].

2. *Drag-Link Mechanism* *Drag link mechanism* is a *quick-return mechanism* having a complete revolution of crank and follower. It is obtained by fixing the shortest link [Fig. 3.6].

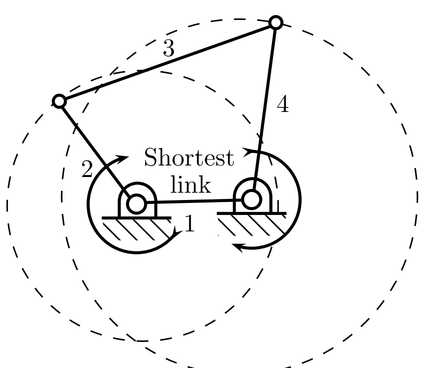


Figure 3.6 | Drag-link mechanism.

3. *Double Crank Mechanism* A *double crank mechanism* consists of two cranks, as seen in following applications:

   (a) *Couple Wheel Locomotive* - In coupled wheel locomotive mechanism, the opposite links are of equal lengths:

$$l_1 = l_3$$
$$l_2 = l_4$$

   (b) *Pantograph* - A pantograph is used to produce a path described by a point either to an enlarged or reduced scale [Fig. 3.7].

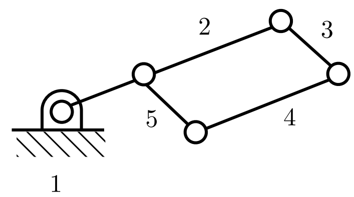


Figure 3.7 | Pantograph.

This mechanism has a peculiarity in that instead of one fixed link, only a point is fixed as a pivot while input motion is given by moving a point on some link along some given planar curve. Applications of pantograph include profile grinding, indicator rig, etc.

   (c) *Parallelogram* - In parallelograms also, the opposite links are of the same length.

3.1.7 Slider Crank Mechanism

A *slider crank mechanism* is a modification of the four-bar chain in which a turning pair is replaced by sliding pair; the mechanism consists of one sliding pair and three turning pairs [Fig. 3.8]. It is used to convert rotary motion into reciprocating motion and vice versa in reciprocating machines.

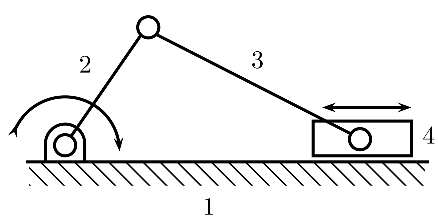


Figure 3.8 | Slider-crank mechanism.

The inversions of slider-crank mechanism and their applications are described as follows:

1. *Frame-Fixed* In this inversion, link 1 (frame) is fixed and an adjacent linkage is made crank. Following are the important applications of this inversion:

(a) Reciprocating engines
(b) Reciprocating compressors

2. *Crank-Fixed* Fixing of link 2 (crank) makes link 3 (connecting rod) to rotate about joint 2-3. The inversion applies in the following applications:

   (a) *Whitworth Quick Return Mechanism* - This mechanism is used in metal cutting machines in which forward stroke takes a little longer and cuts the metal, whereas the return stroke is idle and takes a shorter period [Fig. 3.9].

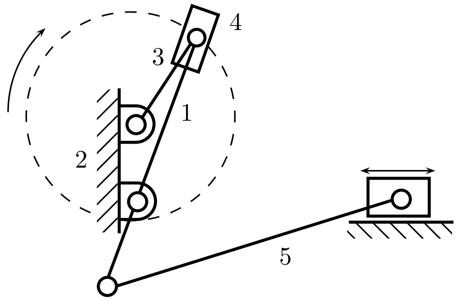


Figure 3.9 | Whitworth quick return motion.

   (b) *Rotary Engine* - This mechanism is obtained by replacing the slider by a piston and making link 1 (frame) to act as pivoted cylinder. Instead of one cylinder, seven, or nine cylinders symmetrically placed at regular intervals in the same plane are used. All the cylinders rotate about the same axis and form a balanced system.

3. *Connecting Rod-Fixed* This mechanism is obtained by fixing the link 3 (connecting rod), which makes link 1 to oscillate about the joint of 2-3. The following are the important applications of this inversion:

   (a) *Oscillating Cylinder Engine* - In this application, the piston reciprocates inside the cylinder pivoted to the fixed link.
   (b) *Slotted Lever-Crank Mechanism* - If the cylinder is made to work as a guide, and the piston in the form of a slider, it results into the slotted lever-crank mechanism [Fig. 3.10].

4. *Slider-Fixed* If link 4 (slider) is fixed, it makes link 3 (connecting rod) to oscillate about fixed pivot 1-2. Inversion by fixing the slider is applied in hand pump.

3.1.8 Double Slider Crank Mechanism

It is possible to replace two turning pairs by two sliding pairs of four-bar mechanism to get a *double slider crank mechanism* [Fig. 3.11].

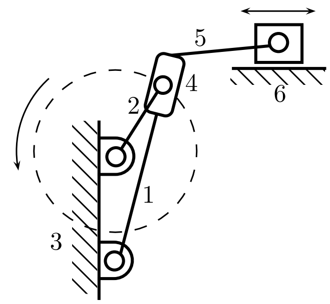


Figure 3.10 | Slotted lever-crank mechanism.

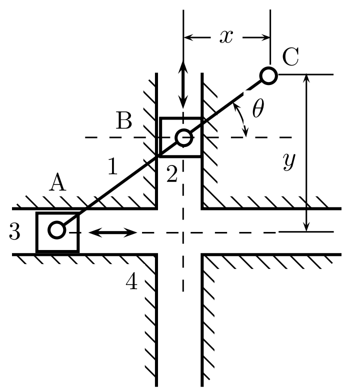


Figure 3.11 | Double slider crank mechanism.

Following are inversions and their applications of this mechanism:

1. *Frame-Fixed* The inversion obtained by fixing the frame is seen in *elliptical trammel* [Fig. 3.11]. With the movement of the sliders, any point on the (connecting) link (3), except mid-point, traces an *ellipse* on a fixed plate. The mid-point of the link 3 traces a circle. To examine, let the link (3) AB makes an angle θ with the x-axis. Considering the displacements of the sliders from the center-line of the trammel,

$$x = \text{BC}\cos\theta$$
$$y = \text{AC}\sin\theta$$

Therefore,

$$\frac{x}{\text{BC}} = \cos\theta$$
$$\frac{y}{\text{AC}} = \sin\theta$$

Squaring and adding

$$\frac{x^2}{\text{BC}^2} + \frac{y^2}{\text{AC}^2} = 1 \tag{3.1}$$

This is an equation of ellipse, indicating that the path traced by point C is an ellipse which has its semi-major and semi-minor axes equal to AC and BC, respectively.

For the special case, when C is the mid-point of AB, AC = BC, and Eq. (3.1) transforms into the equation of a circle of diameter equal to the length of the link.

2. *Slider-Fixed* The inversion achieved by fixing either of the sliders is applied in *scotch yoke* which is used to convert the rotary motion into a sliding motion [Fig. 3.12].

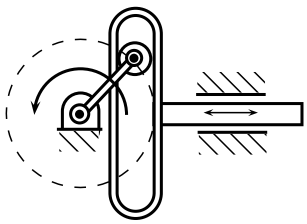


Figure 3.12 | Scotch yoke.

This mechanism is most commonly used in control valve actuators in high pressure oil and gas pipelines. As the connecting link AB rotates like a crank, the horizontal portion of the link (frame) reciprocates in the fixed link (any one of the sliders).

3. *Crank-Fixed* The inversion achieved on crank fixed is applied in *Oldham Coupling* which is used to join two rotating parallel shafts at a small distance [Fig. 3.13].

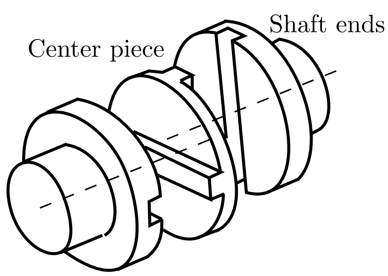


Figure 3.13 | Oldham coupling.

Two flanges, each having a rectangular slot, are keyed, one on each shaft. The two flanges are positioned such that the slot in one is at right angle to the slot in the other. All the rotating elements have the same angular velocity at every instant. The path followed by the center of intermediate piece is circle of diameter equal to distance between the two parallel shafts.

Maximum sliding velocity of the intermediate piece is equal to peripheral velocity of the center of the disc along its circular path (i.e., $c \times \omega$ where c is parallel distance and ω is angular velocity of shafts). Angular velocity of the center of cross about the center of its circle is two times that of angular velocity of the cross.

3.1.9 Degrees of Freedom

The number of *degrees of freedom* or *mobility* of a system is the number of independent variables that must be specified to completely define the condition of the system. Same applies to rigid body, kinematic pair, and kinematic chain, discussed as follows:

1. *Rigid Body* Degree of freedom for a rigid body is defined as the number of possible motions in which the body can move in the given space.

   An unconstrained rigid body in space can be described in six independent motions:

   (a) Translation along x, y, z axes
   (b) Rotation about x, y, z axes

   Thus, a rigid body possesses six degrees of freedom. Connection with other bodies through pairs impose certain constraints on the relative motion, hence, the number of degrees of freedom is reduced.

2. *Kinematic Pair* Degree of freedom of a pair is defined as the number of independent relative motions a pair can have in the given space.

3. *Kinematic Chain* To obtain constrained or definite motions of the links of a mechanism, it is necessary to know how many inputs are required to specify the position of the mechanism. In some mechanisms, only one input is necessary that determines the motion of other links, and it is said to have one degree of freedom. The degree of freedom of a structure is zero. A structure with negative degree of freedom is known as a *super-structure*.

   The following are useful equations to calculate degrees of freedom for kinematic chains.

   (a) *Gruebler's Equation* - A rigid link in a plane has three degrees of freedom; an planar assembly having n links shall posses total degrees of freedom $3n$, before the links are joined together. Each revolutionary pair or joint will remove two degrees of freedom (e.g. x_i, y_i). Thus, if n is number of links of a mechanism including fixed links, f_1 is the number of pin joints or revolute pairs or pairs that permit one degree of freedom (i.e. the number of pint joints plus the number of sliding pairs or total number of lower pairs), then, degrees of freedom, say F, are found as

$$F = 3(n-1) - 2f_1 \tag{3.2}$$

This is called *Gruebler's equation*.

(b) *Kutzbach Equation* - When there are such pairs which remove only one degree of freedom (f_2, number of roll sliding pair or total number of higher pairs), then the Gruebler's equation [Eq. (3.2)] is modified into

$$F = 3(n-1) - 2f_1 - f_2 \tag{3.3}$$

This is called the *Kutzbach equation*. Alternative form of this expression can be obtained in terms of number of binary joints J, number of higher pairs f_2, for n number of links as

$$J + \frac{f_2}{2} = \frac{3}{2}n - 2 \tag{3.4}$$

or

$$n = \frac{2}{3}\left(J + 2 + \frac{f_2}{2}\right) \tag{3.5}$$

In the above equation, one ternary joint is equivalent to two binary joints, and one quaternary joint is equivalent to three binary joints [*Section* 3.1.2].

3.2 UNIVERSAL JOINT

A *universal joint* is used to connect two non-parallel and intersecting shafts and misaligned shafts, for example, to transmit power from the gearbox to rear axle in an automobile. Universal joint is also known as *Hooke's joint*[1].

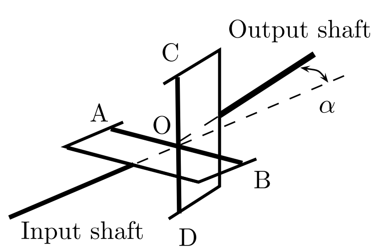


Figure 3.14 Hooke's joint.

Consider a universal joint, connecting two horizontal shafts 1 and 2 at an angle α [Fig. 3.14]. The shafts are supported on the bearings. Each shaft has a fork at its end. The four ends of the two forks of shafts are connected by a center piece, the right angle arms of which rest in the bearings, provided in the fork ends of both shafts.

[1]During 1667-1675, Robert Hooke analyzed the joint and found that its speed of rotation was non-uniform. The first recorded use of the term universal joint for this device was by Hooke.

3.2.1 Shaft Rotations

Let θ be the (absolute) angle rotated by shaft-1 (driver), ϕ be the (absolute) angle rotated by shaft-2 (driven). Using projections of the fork ends, the following relation can be found between the angle moved by the two shafts:

$$\frac{\tan\phi}{\tan\theta} = \sec\alpha \tag{3.6}$$

This is the basic equation for the rotation of shaft-2 with respect to that of shaft-1. This equation will be utilized in deriving expressions for speed ratio and acceleration of the shaft-2.

3.2.2 Shaft Speeds

3.2.2.1 Speed Ratio Differentiating Eq. (3.6) w.r.t. time t

$$\begin{aligned}\sec^2\phi\frac{d\phi}{dt}\cos\alpha &= \sec^2\theta\frac{d\theta}{dt}\\ \frac{\omega_2}{\omega_1} &= \frac{d\phi/dt}{d\theta/dt}\\ &= \frac{\sec^2\theta}{\sec^2\phi\cos\alpha}\\ &= \frac{\cos\alpha}{1-\sin^2\alpha\cos^2\theta}\end{aligned}$$

The following are the three interesting situations where the expression of ω_2/ω_1 can be described with respect to the angular position of shaft-2 (θ):

1. $\omega_2 = \omega_1$:

$$\tan\theta = \pm\sqrt{\cos\alpha} \tag{3.7}$$

2. Minimum ω_2/ω_1:

$$\sin^2\theta = 0$$
$$\theta = 90^\circ, 270^\circ \tag{3.8}$$
$$\left(\frac{\omega_2}{\omega_1}\right)_{\min} = \cos\alpha \tag{3.9}$$

3. Maximum ω_2/ω_1:

$$\cos^2\theta = 1$$
$$\theta = 0^\circ, 180^\circ \tag{3.10}$$
$$\left(\frac{\omega_2}{\omega_1}\right)_{\max} = \frac{1}{\cos\alpha} \tag{3.11}$$

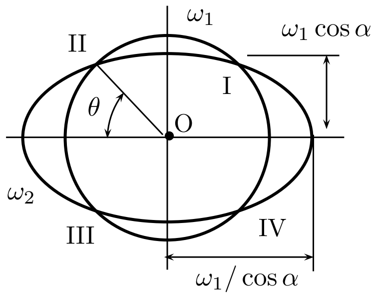


Figure 3.15 Velocities of shafts in Hooke's joint.

3.2.2.2 Speed Variation The variation in the speed of the driven shaft ω_2 w.r.t. ω_1 can be drawn in the *polar velocity diagram* for a given value of α [Fig. 3.15].

In the diagram, ω_1 is an exact circle but ω_2 is an ellipse. The maximum variation in the velocity of driven shaft w.r.t. its mean velocity ω_1 is

$$\begin{aligned}\frac{\omega_{\max}-\omega_{\min}}{\omega_1} &= \frac{1}{\cos\alpha}-\cos\alpha\\ &= \frac{\sin^2\alpha}{\cos\alpha}\\ &= \tan\alpha\sin\alpha\end{aligned}$$

For small values of α in rad,

$$\tan\alpha \approx \sin\alpha \approx \alpha^2$$

therefore,

$$\frac{\omega_{\max}-\omega_{\min}}{\omega_1} \propto \alpha^2$$

3.2.3 Angular Acceleration

Angular acceleration of the driven shaft is

$$\begin{aligned}\dot{\omega}_2 &= \frac{d\omega}{dt}\\ &= \omega_1 \frac{d}{dt}\left(\frac{\cos\alpha}{1-\sin^2\alpha\cos^2\theta}\right)\\ &= \frac{-{\omega_1}^2\cos\alpha.\sin^2\alpha.\sin 2\theta}{\left(1-\sin^2\alpha.\cos^2\theta\right)^2}\end{aligned}$$

For minimum or maximum $\dot{\omega}_2$,

$$\cos 2\theta \approx \frac{2\sin^2\alpha}{2-\sin^2\alpha} \tag{3.12}$$

As is evident in Fig. 3.15, the acceleration is maximum in II and IV quadrants and minimum in I and III quadrants. The acceleration is zero at four values of θ:

$$\begin{aligned}\sin 2\theta &= 0\\ \cos 2\theta &= 1\\ \theta &= 0^\circ, 90^\circ, 180^\circ, 270^\circ\end{aligned}$$

The angle between the two shafts α should be kept as minimum as possible and excessive masses should not be attached to the driven shaft, otherwise, very high alternating stresses due to angular acceleration and retardation will be set up in the parts of the joint, which are undesirable.

3.2.4 Double Hooke's Joints

A *double Hooke's joint* can be obtained by joining two Hooke's joints using an intermediate shaft. If the misalignment between each shaft and the intermediate shaft is equal, the driving and the driven shafts remain in exact angular alignment, though the intermediate shaft rotates with varying speed.

In a double Hooke joint, the angular arrangement of the two Hooke's joints decides the velocity ratio at any instant. It is immaterial whether the driven shaft makes the angle with the axis of driving shaft to its left or right. The velocity ratio depends upon the position of forks, as explained as follows:

1. *Constant Velocity Ratio* The constant velocity ratio is achieved when driving and driven shafts make an equal angle with the intermediate shaft and forks of intermediate shaft lie in the same plane. This is the reason why double hook joints are used, because then dynamic stresses are reduced to zero.

2. *Varying Velocity Ratio* If the above condition is not satisfied, then the speed ratio varies between maximum and minimum values, given by

$$\begin{aligned}\left(\frac{\omega_2}{\omega_1}\right)_{\max} &= \frac{1}{\cos^2\alpha}\\ \left(\frac{\omega_2}{\omega_1}\right)_{\min} &= \cos^2\alpha\end{aligned}$$

Double Hooke's joints are used in connecting two drive-shafts.

3.3 KINEMATIC ANALYSIS

Each *kinematic link* of a machine has a relative motion in a definite path, which can be either straight, circular or curved. Kinematic analysis usually aims at determining motion characteristics of various links in a mechanism. Such information is essential for computing forces and thereby dimensions of the links, enabling design of various links in a mechanism.

Two types of methods are available for kinematic analysis: graphical method and analytical method. Graphical methods are essential in developing a conceptual understanding about the subject. They provide the fastest method of checking the results, though less accurate.

3.3.1 Velocity of a Link

In kinematic analysis, all the motions are measured relative to some reference axes or planes. Usually, the earth is taken to be a fixed reference plane, and all such motions relative to it are termed as *absolute motion*. When motion of a body is measured with respect to another body, in motion or steady state, it is called *relative motion*. In mechanism, the motion of a link can be measured with respect to fixed points as well as moving points on the links.

3.3.1.1 Linear Velocity Let a rigid link OA of length r rotate about a fixed point O with a uniform velocity ω. In a small interval of time δt, the link turns through a small angle $\delta\theta$ and the point A moves to new location A′ [Fig. 3.16]. Displacement of point A equal to $r\delta\theta$,

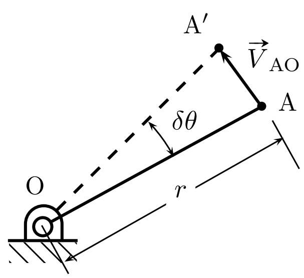


Figure 3.16 Linear velocity of a link.

therefore, *linear velocity* of the point A relative to point O is defined as

$$V_{\mathrm{AO}} = \lim_{\delta t \to 0} \frac{r\delta\theta}{\delta t}$$
$$= r \lim_{\delta t \to 0} \frac{\delta\theta}{\delta t}$$
$$= r\omega_{\mathrm{AO}} \tag{3.13}$$

where ω_{AO} represents the angular velocity of the link, given by

$$\omega_{\mathrm{AO}} = \lim_{\delta t \to 0} \frac{\delta\theta}{\delta t} \tag{3.14}$$

The direction of V_{AO} is along the displacement of A. When $\delta t \to 0$, AA′ will be perpendicular to OA. This emerges from the fact that A can neither approach nor recede from O and thus, the only possible motion of A relative to O is in a direction perpendicular to OA.

The magnitude of the instantaneous linear velocity V_{AO} ($=r\omega_{\mathrm{AO}}$) of a point on a rotating body is proportional to its distance from the axis of rotation:

$$V_{\mathrm{AO}} \propto r$$

Therefore, velocity of an intermediate point B on the link OA [Fig. 3.17] can be found as

$$\frac{V_{\mathrm{BO}}}{V_{\mathrm{AO}}} = \frac{\mathrm{BO}}{\mathrm{AO}}$$

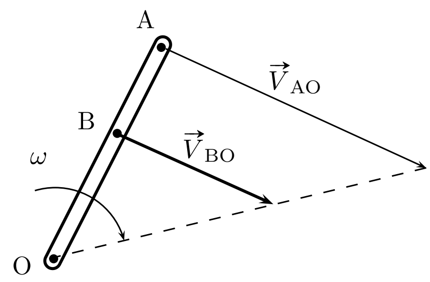


Figure 3.17 Velocity of an intermediate point.

Consider a case of link AB in which absolute velocities of points A and B are $\vec{V}_{\mathrm{A}}$ and $\vec{V}_{\mathrm{B}}$, respectively [Fig. 3.18]. The *relative velocity* of point B with respect to A is given by

$$\vec{V}_{\mathrm{BA}} = \vec{V}_{\mathrm{B}} - \vec{V}_{\mathrm{A}}$$

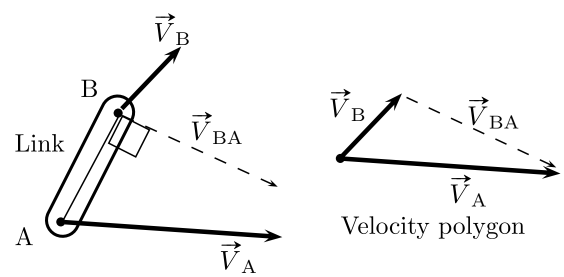


Figure 3.18 Relative velocity in a link.

If $\vec{V}_{\mathrm{A}}$ is completely known, but $\vec{V}_{\mathrm{B}}$ is known in direction (the tangent to the path followed B) only, then magnitude of $\vec{V}_{\mathrm{B}}$ is also determined because direction of $\vec{V}_{\mathrm{BA}}$ will be perpendicular to the link AB. This observation is used in graphical methods.

A *vector polygon* is a graphical depiction of vector equations of velocities of two or more points in a link or mechanism. It contains information about the magnitude and direction of velocities [Fig. 3.18].

3.3.1.2 Angular Velocity *Angular velocity* of a link is defined as the linear velocity divided by its radius. Similar to Eq. (3.14), in reference to the previous case of Fig. 3.18, angular velocity of link AB is given by

$$\omega_{\mathrm{AB}} = \frac{V_{\mathrm{AB}}}{\mathrm{AB}}$$

3.3.1.3 Instantaneous Center For two bodies having relative motion with respect to one another, instantaneous center of rotation is an imaginary point common to the two bodies such that any of the two bodies can be assumed to have motion of rotation with respect to the other about the imaginary point. In general, instantaneous center of rotation is not a stationary point since as the body moves from one position to another, the velocities of their points keep changing.

Let a point A on a rigid body have a linear velocity $\vec{V}_A$ and the body itself has an angular velocity $\vec{\omega}$ [Fig. 3.22]. These two quantities completely define the velocities of all points or particles in the rigid body. To appreciate this sentence, let a perpendicular be erected at point A to $\vec{V}_A$ and the distance $r_A = V_A/\omega$, measured along it to locate instantaneous center 'I' of the body [Fig. 3.19].

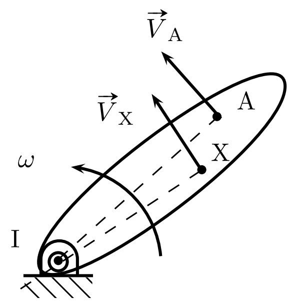


Figure 3.19 | Instantaneous center.

The magnitude of linear velocity of any other point X, at any given instant, will then be given as

$$V_X = \text{IX} \times \omega$$

Based on the above equation, the instantaneous center (I) is the intersection point of the normals of velocities at any two points in the body.

Instantaneous center between the two links (say, 2 and 3) is denoted by I_{23}.

3.3.1.4 Number of Instantaneous Centers In a mechanism, the number of instantaneous centers is the number of possible combinations of two links. A mechanism, having n links, will have the number of instantaneous centers given by

$$N = {}^nC_r = \frac{n\,(n-1)}{2}$$

Arnold Kennedy's theorem states that three bodies, having relative motion with respect to one another, have three instantaneous centers, all of which lie on the same straight line.

When extended to kinematic chains, Arnold Kennedy's theorem implies that with every combination of three links, there are three I's and if two of them are known, the third one will lie on the line joining them. This concept can be used to locate the instantaneous centers of mechanism, for example, four-bar mechanisms [Fig. 3.20] and slider-crank mechanism [Fig. 3.21].

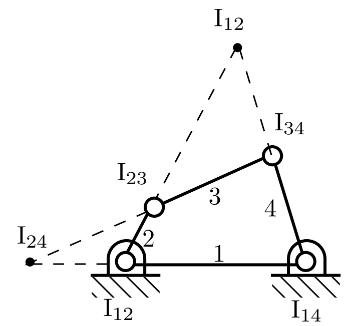


Figure 3.20 | Four-bar mechanisms.

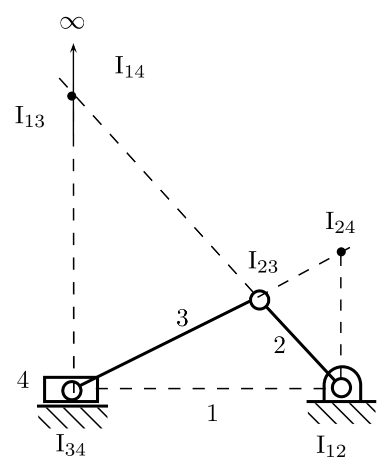


Figure 3.21 | Slider-crank mechanism.

Few simple examples of the location of the instantaneous centers are illustrated in Fig. 3.22.

3.3.2 Acceleration in Mechanism

Acceleration is an important aspects in the design of mechanisms as it directly indicates the inertia forces of the members. *Linear acceleration* is defined as the rate of change of *linear velocity* with respect to time:

$$\vec{a} = \frac{d\vec{V}}{dt}$$

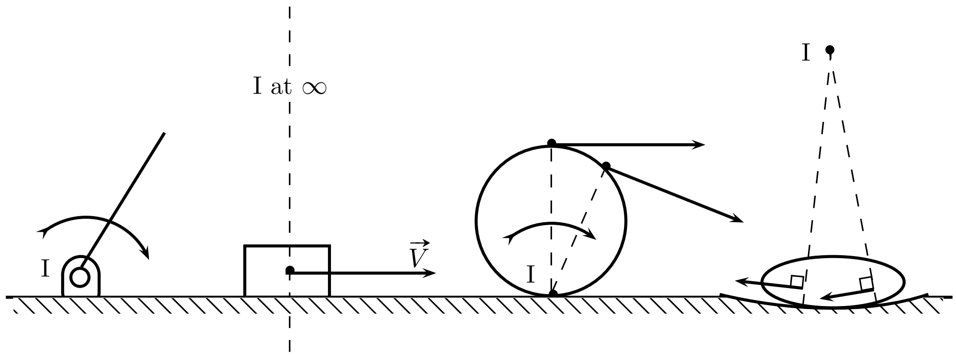


Figure 3.22 Examples of locating I.

Similarly, *angular acceleration* is defined as the rate of change of angular velocity with respect to time,

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt}$$

Angular acceleration of the particle, having a circular motion of radius r and angular velocity ω, is given by

$$\alpha = r\omega^2$$

If a link OA moves with angular velocity ω_{AO} [Fig. 3.13] then the angular acceleration of point A on the link w.r.t. O is given by

$$\alpha_{\text{AO}} = \text{AO} \times \omega_{\text{OA}}{}^2$$

When a slider moves over a link which itself is moving, the acceleration of the slider involves an important component known as *Coriolis component*[2]. It is the tangential component of acceleration of a slider with respect to the coincident point on the link. To examine this, let a link OA rotate about a fixed point O. Let ω, α represent the angular velocity, angular acceleration of OA, respectively, v, f represent linear velocity and linear acceleration of a point P on the slider of the link at radial distance r from the center O [Fig. 3.23]. The components of acceleration of a point P on the slider can be determined as follows:

1. Acceleration Along OA The change in velocity of point P along the rotating link OA is given by

$$\delta v_P^{\parallel} = \{(v+f\delta t)\cos\delta\theta - (\omega+\alpha\delta t)r'\sin\theta\} - v$$

With limiting case of $\delta t \to 0$, $\cos\delta\theta \to 1$, $\sin\delta\theta \to 0$. Hence, acceleration of P along OA is

$$a_P^{\parallel} = \lim_{\delta t\to 0}\frac{\delta v_P^{\parallel}}{\delta t} = f - \omega^2 r \tag{3.15}$$

[2]The mathematical expression for the Coriolis component appeared in an 1835 paper by French scientist Gaspard-Gustave Coriolis.

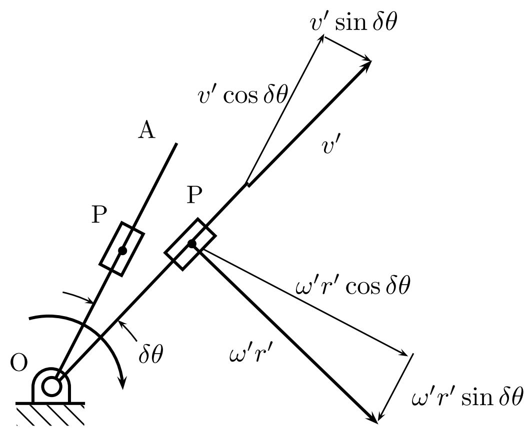


Figure 3.23 Coriolis component.

2. Acceleration Normal to OA The change in velocity of point P perpendicular to the rotating link OA is

$$\delta v_P^{\perp} = (v+f\delta t)\sin\delta\theta + (\omega+\alpha\delta t)(r+\delta r)\cos\delta\theta - \omega r$$

With limiting case of $\delta t \to 0$, $\cos\delta\theta \to 1$, $\sin\delta\theta \to 0$. Hence, acceleration of P perpendicular to OA as

$$a_P^{\perp} = \lim_{\delta t\to 0}\frac{\delta v_P^{\perp}}{\delta t} = 2\omega v + r\alpha \tag{3.16}$$

In this equation, the component $2\omega v$ is known as *Coriolis component*. The remaining component is the tangential acceleration.

The total acceleration of the point P would be the vector sum of the orthogonal components of acceleration, given by Eqs. (3.15) and (3.16):

$$a_P^{\parallel} = f - \omega^2 r$$
$$a_P^{\perp} = 2\omega v + r\alpha$$

The *Coriolis component* of acceleration exists in mechanisms having a slider in a rotating link with angular acceleration, for example, quick-return mechanism.

3.4 CAM FOLLOWER MECHANISM

A *cam* mechanism usually consists of two moving elements, *cam* and *follower*, mounted on a fixed frame. A *cam* can be defined as a machine element having a curved outline or a curved groove, which, by its oscillation or rotation motion, gives a predetermined specified motion to another element known as *follower*. In a cam-follower mechanism, the follower usually has a line-contact with the cam, thus, they constitute a higher pair mechanism.

Cam devices are versatile through which almost any arbitrarily-specified motion can be obtained. In some instances, they offer the simplest and most compact way to transform motions. Cams are widely used in automatic machines, internal engines, machine tools, printing control mechanisms, and so on.

3.4.1 Types of Cams

Different types of cams are shown in Fig. 3.24. These are explained as follows:

1. Wedge Cam A *wedge cam* consists of a wedge which in general has a translation motion. The contact between the cam and follower is maintained by using a spring.
2. Disc Cam In *disc cams*, the follower moves radially from the center of rotation of the cam; the transmission line passes through center of cam.
3. Spiral Cam A *spiral cam* contains a spiral groove which mesh with a pin-gear follower. Rotation of the cam is reversed to reset the the follower.
4. Cylindrical Cam A *cylindrical cam* is a cylinder having a circumferential contour in the surface, which mesh with a pin-gear follower.
5. Conjugate Cam A *conjugate cam* is made of two disc cams, keyed together in such way that makes a positive constraint in touch with two rollers of a follower. These cams are preferred due to low wear and noise in high speed, and high dynamic load.
6. Globoidal Cams A *globoidal cam* is similar to cylindrical cam but it can have convex or concave surface with a circumferential contour to impart oscillatory motion to the follower.
7. Spherical Cam In a *spherical cam*, the follower oscillates about an axis perpendicular to the axis of rotation of cam having a spherical surface.

3.4.2 Types of Followers

Different types of followers are shown in Fig. 3.25. These

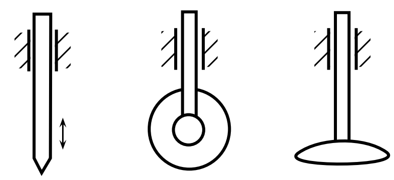


Figure 3.25 Types of cam followers.

are explained as follows:

1. Knife Edge Follower *Knife edge followers* are not used mostly due to great wear at knife edge and considerable side thrust.
2. Roller Follower In the case of steep rise, a *roller follower* jams the cam and hence is not preferred for such design.
3. Mushroom Follower *Mushroom followers* are of two types: flat faced and spherical faced. There is no side thrust except due to friction at the contact of cam and the follower.

3.4.3 Terminology

The following are the important design dimensions and geometries associated with the cam and follower [Fig. 3.26].

1. Basic Circle The smallest circle to the cam profile and concentric to the center of rotation of the cam is known as *base circle*.
2. Trace Point The *trace point* is the point of tracing on the follower. It is actually the edge of knife edge follower or the center of roller follower.
3. Prime Curve The curve drawn by the trace point by fixing the cam at an angle is called *prime curve*.
4. Prime Circle The smallest circle to pitch curve is called *prime circle*.

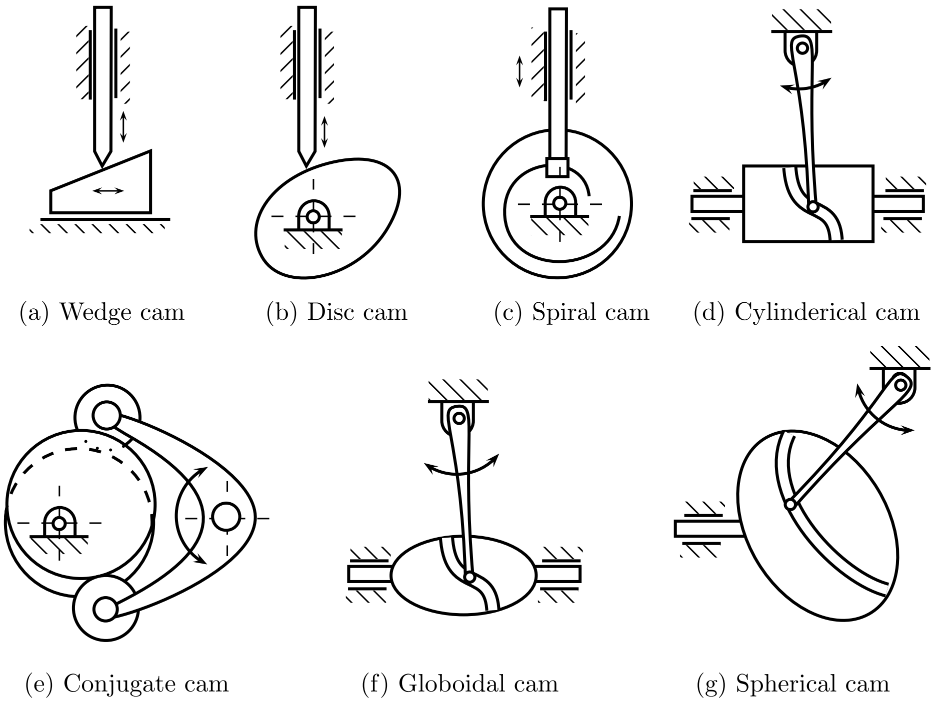


Figure 3.24 Types of cams.

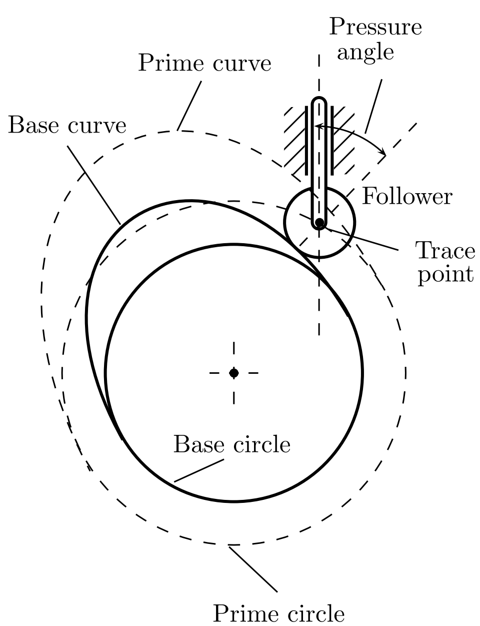


Figure 3.26 Terminology of cam-follower.

5. <u>*Pressure Angle*</u> *Pressure Angle* is the angle between normal to the pitch curve at a point and direction of motion of follower.

For the same rise or fall as the pressure angle increases size (or base circle) of cam decreases. The size of base circle controls the pressure angle for given rise. The increase in the base circle diameter increases the length of the arc of the circle upon which the wedge (the raised portion) is to be made.

6. <u>*Pitch Point*</u> *Pitch point* is the point on a cam profile for which the *pressure angle* attains maximum value.

7. <u>*Pitch Circle*</u> The circle passing through pitch point and concentric to the center of rotation of the cam is called *pitch circle*.

3.4.4 Motion of the Follower

As a cam rotates about its axis, it imparts a specific motion to the follower which repeats in each revolution of the cam. The position of the follower, say measured from its lowest position, depends upon the angular position of cam:

$$s = f(\theta) \tag{3.17}$$

This relation can be visualized by plotting the angular displacement (θ) of the cam on x-axis and the linear displacement (s) of the follower on y-axis.

The function $s = f(\theta)$ represents the design scheme of the displacement of the follower. In this relation, the following terms are used to describe various elements of the motion of the cam and the follower:

1. *Elements of Motion of Cam*
   (a) *Angle of Ascent* - *Angle of ascent* (ϕ_a) is the angle through which the cam turns during the time the follower rises.
   (b) *Angle of Dwell* - *Angle of dwell* (δ) is the angle through which the cam turns while the follower remains stationary at the highest or the lowest position.
   (c) *Angle of Descent* - *Angle of descent* (ϕ_d) is the angle through which the cam turns during the time the follower returns to the initial position.
   (d) *Angle of Action* - *Angle of action* is the total angle moved by the cam during the time between the beginning of rise and end of return of the follower.
2. *Elements of Motion of the Follower*
   (a) *Lift* - *Lift* is the maximum displacement of the follower.
   (b) *Ascent* - *Ascent* or *rise* is the movement of the follower away from the center of cam.
   (c) *Dwell* - *Dwell* is the period when there is no movement of the follower. In internal combustion engines, a shorter dwell period means a smaller period of valve opening, resulting in less fuel per cycle and lesser power production. Thus, the minimum value of dwell angle cannot be reduced from a certain value.

Using Eq. (3.17), the profiles of velocity and acceleration of the follower motion can also be determined as

$$v = \frac{ds}{dt}$$
$$f = \frac{dv}{dt}$$

Angular speed of cam is represented by

$$\omega = \frac{d\theta}{dt}$$

The following sections deal with the basic displacement functions of the ascent or descent of the follower. The ascent or descent of the follower takes place when the cam rotates an angle ϕ. The lift is represented by h.

3.4.4.1 Simple Harmonic In simple harmonic profile, the ascent and descent of the follower takes place in a half cycle (π) of the sinusoidal function. Therefore, half cycle of the harmonic functions is equivalent to the rotation cam (2ϕ).

The ascent takes place when cam rotates angle ϕ. Therefore, the simple harmonic displacement (s) can be related to angular displacement ($\theta = 0 \rightarrow \phi$) as

$$s = \frac{h}{2}\left\{1 - \cos\left(\frac{\pi\theta}{\phi}\right)\right\}$$

Velocity function is found as

$$V = \frac{ds}{dt} = \frac{h}{2}\left\{\frac{\pi}{\phi}\omega \sin\left(\frac{\pi\theta}{\phi}\right)\right\}$$

Acceleration function is found as

$$f = \frac{dv}{dt} = \frac{h}{2}\left\{\frac{\pi^2}{\phi^2}\omega^2 \cos\left(\frac{\pi\theta}{\phi}\right)\right\}$$

The functions of s, v and f are drawn in a combined plot [Fig. 3.27]. The velocity increases from zero to maximum at quarter of the simple harmonic motion;

$$\frac{\pi\theta}{\phi} = \frac{\pi}{2}$$
$$\theta = \frac{\phi}{2}$$

Therefore, the maximum value of velocity is given by

$$v_{\max} = \frac{h}{2}\left(\frac{\pi\omega}{\phi}\right)$$

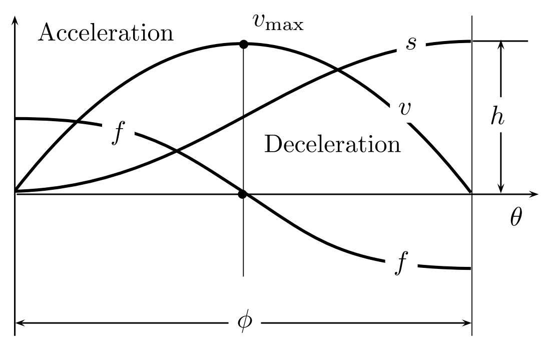


Figure 3.27 Simple harmonics.

At the beginning of rise ($\theta = 0$), acceleration (f) of the follower suddenly increases from zero to maximum value, given by

$$f_{\max} = \frac{h}{2}\left(\frac{\pi\omega}{\phi}\right)^2$$

The maximum acceleration at the beginning itself causes higher inertia loads of the follower, therefore, the simple harmonic motion of the follower is usually applied for moderate speeds only.

3.4.4.2 Constant Acceleration A *constant acceleration* profile gives a constant acceleration in the first half of the ascent and a constant deceleration in the second of the ascent. Consider a case when the velocity of follower at the beginning of the rise is zero ($v_0 = 0$). At any instant of time t taken from the beginning, the displacement s during ascent can be related to acceleration f using the second equation of linear motion:

$$\begin{aligned} s &= v_0 t + \frac{1}{2} f t^2 \\ &= \frac{1}{2} f t^2 \end{aligned} \tag{3.18}$$

This is an equation of parabola, thus, the profile of constant acceleration is also known as *parabolic motion*. Since there a constant acceleration in the first half of the ascent and a constant deceleration in the second half of the ascent, the ascents in the first and the second halves will be the same.

The arbitrary time t can be related to angular position θ of the cam as

$$t = \frac{\theta}{\omega}$$

The ascent and descent of the follower takes place when the cam rotates an angle 2ϕ; ascent upto lift h takes place when cam rotates an angle ϕ. For the constant angular speed ω, the cam takes time ϕ/ω for lift h of the follower. Taking the lift upto to the mid-point where the acceleration changes its sign and using Eq. (3.18),

$$\begin{aligned} \frac{h}{2} &= \frac{1}{2} f \left(\frac{\phi}{2\omega}\right)^2 \\ f &= 4h \left(\frac{\omega}{\phi}\right)^2 \end{aligned}$$

Putting these values in Eq. (3.18), the displacement function is found as

$$\begin{aligned} s &= \frac{1}{2} \times 4h \left(\frac{\omega}{\phi}\right)^2 \left(\frac{\theta}{\omega}\right)^2 \\ &= \frac{2h}{\phi^2}\theta^2 \end{aligned}$$

Therefore, the function of velocity is found as

$$\begin{aligned} v &= \frac{ds}{dt} \\ &= \frac{4h\omega}{\phi^2}\theta \end{aligned}$$

This equation represents a linear profile of the velocity, having constant slope, equal to acceleration f. The functions s, v, and f are drawn in a combined plot [Fig. 3.28].

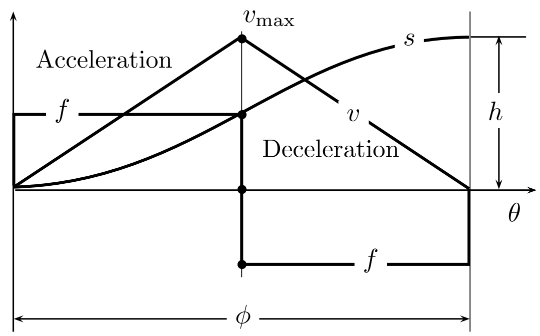


Figure 3.28 Constant acceleration motion.

Velocity is maximum when the acceleration changes its sign ($\theta = \phi/2$):

$$v_{max} = \frac{2h\omega}{\phi} \tag{3.19}$$

At mid-way, an infinite jerk is produced, hence this profile of the follower motion is adopted only upto moderate speeds.

3.4.4.3 Constant Velocity *Constant velocity* of the follower implies that the displacement of the follower is proportional to the cam displacement and the slope of the displacement curve is constant:

$$s \propto h\frac{\theta}{\phi} \tag{3.20}$$

Therefore,

$$\begin{aligned} v &= \frac{ds}{dt} \\ &= \frac{h\omega}{\phi} \end{aligned}$$

Since the velocity function is constant, the acceleration throughout the lift is zero. The functions s, v and f are drawn in a combined plot [Fig. 3.29].

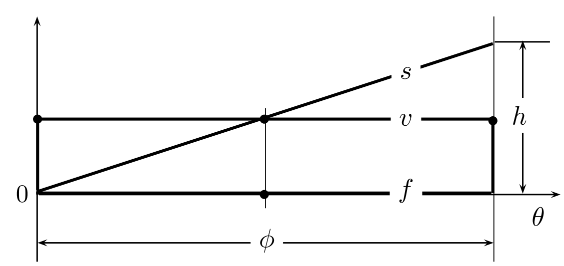


Figure 3.29 Constant velocity profile.

The acceleration of the follower during rise and return period is zero but it is infinite at the beginning and end of the motion due to abrupt changes in the velocity. On account of this reason, the constant velocity profile is generally not used in practice.

3.4.4.4 Cycloidal Profile A *cycloid* is the locus of a point on a circle rolling on a straight line. Following is the relation of rise (s) with cam rotation (θ):

$$s = \frac{h}{\pi}\left(\frac{\pi\theta}{\phi} - \frac{1}{2}\sin\frac{2\pi\theta}{\phi}\right)$$

Therefore, the functions of velocity and acceleration are found as

$$\begin{aligned} v &= \frac{ds}{dt} \\ &= \frac{h}{\pi}\left(\frac{\pi\omega}{\phi} - \frac{\pi\omega}{\phi}\cos\frac{2\pi\theta}{\phi}\right) \\ &= \frac{h\omega}{\phi}\left(1 - \cos\frac{2\pi\theta}{\phi}\right) \\ f &= \frac{dv}{dt} \\ &= \frac{2h\pi\omega^2}{\phi^2}\sin\frac{2\pi\theta}{\phi} \end{aligned}$$

The functions s, v and f are drawn in a combined plot [Fig. 3.30]. The velocity is maximum at mid-point of the lift ($\theta = \phi/2$)

$$v_{\max} = \frac{2h\omega}{\phi} \tag{3.21}$$

At this point, the acceleration is zero.

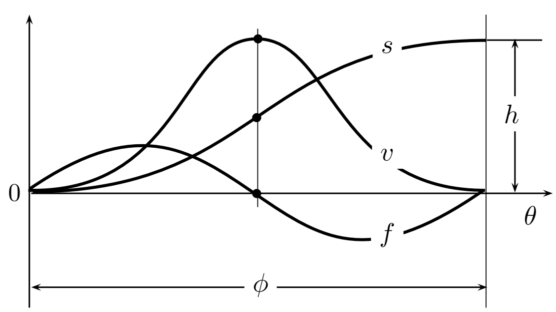


Figure 3.30 Cycloid profile motion.

Maximum value of acceleration occurs at $\theta = \phi/4$:

$$f_{\max} = \frac{2h\pi\omega^2}{\phi^2}$$

There are no abrupt changes in velocity and acceleration, hence, cycloidal program is the most suitable one for high speed follower motion.

3.5 GEARS

If power transmitted between two shafts is small, motion between them can be obtained by using plain cylinders or discs, called *friction wheels*. However, as the power increases, slip occurs between the discs and the motion no longer remains definite. Therefore, *gears* are used to transmit motion from one shaft to another by successive engagements of *teeth* without any intermediate link or connector. In this method, the surfaces of two bodies make a tangential contact and have a rolling motion along the tangent at the point of contact. No motion is possible along the common normal as that will either break the contact or one body will tend to penetrate into the other.

3.5.1 Classification

Gears can be broadly classified according to the relative positions of their shaft axes, discussed under following headings.

3.5.1.1 Gears for Parallel Shafts The motion between two parallel shafts is equivalent to the rolling of two cylinders, assuming no slipping. Gears under this group are the following:

1. *Spur Gears* Straight *spur gears* are the simplest form of gears having straight teeth parallel to the gear axis. The contact of two teeth takes place over the entire width along a line parallel to the axes of rotation. As the gears rotate, the line of contact goes on shifting parallel to the shaft.

   Although there is no axial thrust, but there is a sudden application of load, associated with high impact stresses and excessive noise at high speeds.

2. *Helical Gears* In *helical gears*, the teeth are part of helix instead of straight across the gear parallel to the axis. The mating gears have the same helix angle, but in opposite direction. As the gear rotates, the contact shifts along the line of contact in involute helicoid across the teeth.

   Load application is gradual, because at the beginning of engagement, the contact occurs at the point of leading edge of the curved teeth. Helical gears are, therefore, used at higher velocities and can carry higher loads compared to straight spur gears.

   The inclined direction of forces on the teeth in helical gears results axial thrust.

3. *Double Helical Gears* To avoid the problem of axial thrust in helical gears, *double helical gears* are made of two helical gears with opposite helix angles, which can be up to 45°.

If the left helix and right helix of a double helical gear meet at a common apex and there is no groove in between two pairs of gears, the gear is called *herringbone gear*.

4. Rack and Pinion In this case, the spur rack can be considered to be a spur gear of infinite pitch radius with its axis of rotation placed at infinity parallel to that of pinion. The pinion rotates while the rack translates. This mechanism is used to convert either rotary motion into linear motion or vice versa.

3.5.1.2 Gears for Intersecting Shafts The motion between two intersecting shafts is equivalent to the rolling of two cones, assuming no slipping. Therefore, the gears used for intersecting shafts are called *bevel gears*. The following are their types:

1. Straight Bevel Gears *Straight bevel gears* are provided with straight teeth, radial to the point of intersection of the shaft axes and vary in cross-section through the length inside the generator of the cone. Their main application is in connecting low speed shafts at right angles. In such applications, straight bevel gears of the same size are known as *miter gears*.

   Straight bevel gears can be viewed as a modified version of straight spur gears in which the teeth are made in conical direction instead of parallel to the axis. Also, like in straight spur gears, the line of contacts of straight bevel gears is a straight line. These gears become noisy at higher speed.

2. Spiral Bevel Gears To avail high speed applications, bevel gears are made such that their teeth are inclined at an angle to the face of the bevel, and then they are known as spiral bevel gears or helical bevels.

   Spiral bevel gears are quieter in action but are subjected to axial thrust. These gears find application in the differential drive of automobiles.

3. Zero Bevel Gear *Zero bevel gears* are made with zero spiral angle. Their curved teeth produces quieter motion.

3.5.1.3 Gears for Skew Shafts The following gears are used to join two skew (non-parallel and non-intersecting) shafts:

1. Hypoid Gears The *hypoid gears* are made of the frusta of *hyperboloids of revolution*. Two matching hypoid gears are made by revolving the same line which is in fact their line of contact, therefore, these gears are not interchangeable.

   The relative motion between these gears consists partly of rolling and partly of sliding, along the common line of contact.

2. Worm Gears *Worm gears* are used to connect skewed shafts (i.e. non-parallel and non-intersecting), but not necessarily at right angles. Teeth on worm gear are cut continuously like the threads on a screw. The gear meshing with the worm gear is known as *worm wheel* and the combination is known as worm and worm wheel. At least one teeth of the worm must make a complete turn around the pitch cylinder, and thus forms screw thread.

   Unlike with ordinary gear trains, the direction of transmission in worm drive is not reversible when using large reduction ratios. Due to the greater friction involved between the worm and worm-wheel, usually a single start (one spiral) worm is used.

   If a multi-start worm (multiple spirals) is used then the ratio reduces accordingly and the braking effect of a worm and worm-gear would need to be discounted as the gear will be able to drive the worm.

3.5.2 Gear Terminology

The following are important dimensions and geometries concerned with toothed gears [Fig. 3.31]:

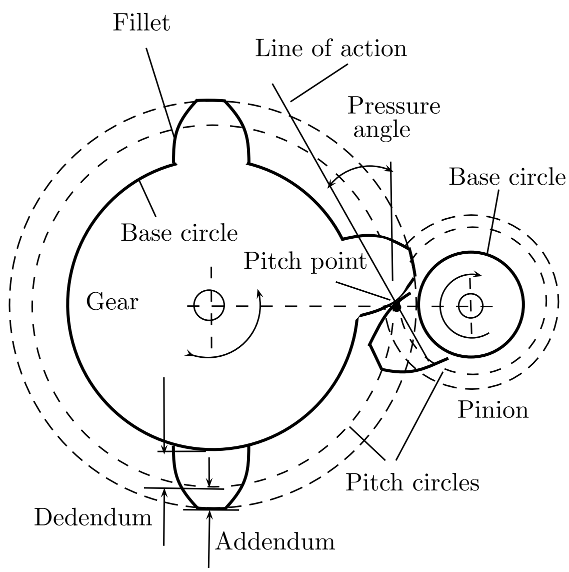


Figure 3.31 Gear terminology.

1. Pitch Circle *Pitch circle* is the apparent circle that two gears can be taken like smooth cylinders rolling without slipping.

2. Addendum Circle *Addendum!circle* is the outermost profile circle of a gear. *Addendum* (a) is the

radial distance between the pitch circle and the addendum circle.

3. Dedendum Circle *Dedendum!circle* is the innermost profile circle. *Dedendum* is the radial distance between the pitch circle and the dedendum circle.

4. Clearance *Clearance* is the radial distance from the top of the tooth to the bottom of the tooth space in the mating gear.

5. Backlash *Backlash* is the tangential space between teeth of mating gears at pitch circles.

6. Full Depth *Full depth* is the sum of the dedendum and the addendum.

7. Face Width *Face width* is the length of tooth parallel to axes.

8. Diametral Pitch *Diametral pitch* (p) is the number of teeth per unit diameter.

   If d is diameter of pitch circle of the gear with T number of teeth, then diametral pitch is calculated as

$$p = \frac{T}{d}$$

9. Module *Module* (m) is the inverse of diametral pitch:

$$m = \frac{1}{p} = \frac{d}{T}$$

10. Circular Pitch *Circular pitch* is the space in pitch circle used by each teeth:

$$\begin{aligned} p_c &= \frac{\pi d}{T} \\ &= m\pi \\ &= \frac{\pi}{p} \end{aligned}$$

11. Gear Ratio *Gear Ratio* (G) is the ratio of numbers of teeth of larger gear to smaller gear.

12. Pressure Line *Pressure line* is the common normal at the point of contact of mating gears along which the driving tooth exerts force on the driven tooth. It is also called *line of action*.

13. Pressure Angle *Pressure angle* (ϕ) is the angle between the pressure line and common tangent to pitch circles. It is also called *angle of obliquity*. High pressure angle requires wider base and stronger teeth.

14. Pitch Angle *Pitch angle* is the angle captured by a tooth. If there are T teeth in a gear, the pitch angle is determined as

$$\text{Pitch angle} = \frac{360^\circ}{T} \tag{3.22}$$

15. Contact Ratio *Contact ratio* is the ratio of angle of action and pitch angle:

$$\text{Contact ratio} = \frac{\text{Angle of action}}{\text{Pitch angle}} \tag{3.23}$$

16. Path of Approach *Path of approach* is the distance along the pressure line traveled by the contact point from the point of engagement to the pitch point.

17. Path of Recess *Path of recess* is the distance along the pressure line traveled by the contact point from the pitch point to the point of disengagement.

18. Path of Contact *Path of contact* is the sum of path of approach and path of recess.

19. Arc of Approach *Arc of approach* is the distance traveled by a point on either pitch circle of the two wheels from the point of engagement to the pitch.

20. Arc of Recess *Arc of recess* is the distance traveled by a point on either pitch circle of the two wheels from the pitch point to the point of disengagement.

21. Arc of Contact *Arc of contact* is the distance traveled by a point on either pitch circle of the two wheels during the period of contact of a pair of teeth.

22. Angle of Action *Angle of action* is the angle turned by a gear during arc of contact.

3.5.3 Law of Gearing

Consider two rigid bodies 1 and 2, representing a portion of the two gears in mesh, rotate about the centers O_1 and O_2, respectively [Fig. 3.32]. The common tangent TT and common normal NN pass through contact point C or C_1 and C_2, the points of contact on respective bodies.

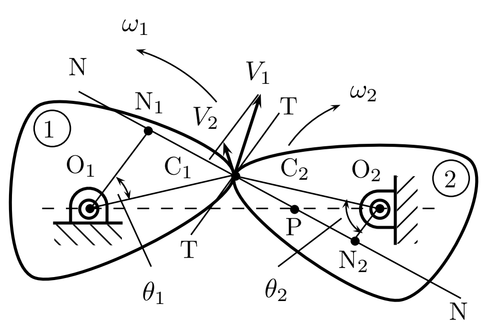


Figure 3.32 | Law of gearing.

The components of relative velocity of the two gears can be determined as follows:

1. Along the Common Normal Relative velocity along the common normal N-N is given by

$$V_1 \cos\theta_1 - V_2 \cos\theta_2 = 0$$
$$\omega_1 \mathrm{O_1C_1} \cos\alpha - \omega_2 \mathrm{O_2C_2} \cos\beta = 0$$
$$\omega_1 \mathrm{O_1N_1} - \omega_2 \mathrm{O_2N_2} = 0$$

Therefore,

$$\frac{\omega_1}{\omega_2} = \frac{\mathrm{O_2N_2}}{\mathrm{O_1N_1}} = \frac{\mathrm{N_2P}}{\mathrm{N_1P}} = \frac{\mathrm{O_2P}}{\mathrm{O_1P}} \tag{3.24}$$

Hence, the angular velocities of the two gears shall remain constant if the common normal at the point of contact of the two teeth passes through a fixed point P which divides the line of centers in the inverse ratio of angular velocities of two gears. This is known as *law of gearing* and the point P is called the *pitch point*.

2. Along the Common Tangent The relative velocity of the mating bodies along the common tangent (TT) at the point of contact is called *velocity of sliding*:

$$\begin{aligned} V_s &= V_1 \sin\theta_1 - V_2 \sin\theta_2 \\ &= \omega_1 \mathrm{O_1C_1}\frac{\mathrm{C_1N_1}}{\mathrm{O_1C_1}} - \omega_2 \mathrm{O_2C_2}\frac{\mathrm{C_2N_2}}{\mathrm{O_2C_2}} \\ &= \omega_1 (\mathrm{N_1P} - \mathrm{C_1P}) - \omega_2 (\mathrm{N_2P} - \mathrm{C_2P}) \\ &= (\omega_1 + \omega_2)\,\mathrm{CP} + (\omega_1 \mathrm{N_1P} - \omega_2 \mathrm{N_2P}) \end{aligned}$$

Applying the *law of gearing* [Eq. (3.24)],

$$V_s = \mathrm{CP} \times (\omega_1 + \omega_2) \tag{3.25}$$

At the pitch point, CP = 0, thus, the velocity of sliding is also zero.

3.5.4 Teeth Profiles

The profile of teeth must satisfy the *law of gearing* so that the mating gears can run without any breakage. There are two types of teeth profiles:

1. *Cycloidal profile*
2. *Involute profile*

Involute profile is preferred due to favorable features, and is discussed later. A gear of one type of teeth can mate only with the gear of same type of teeth, otherwise law of gearing will not be satisfied.

3.5.4.1 Cycloidal Teeth Cycloidal profile is composed of two types of profiles [Fig. 3.33]:

1. *Hypocycloid profile*[3] - inside the pitch curve
2. *Epicycloid profile*[4] - outside the curve.

These curves are produced when a small generating circle rolls inside and outside the perimeter of the pitch circle of the two mating gears. Therefore, cycloidal teeth are also called *double curve teeth* [Fig. 3.33].

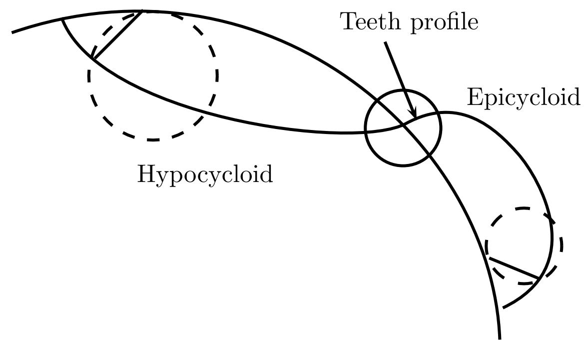


Figure 3.33 | Cycloidal profile.

The cycloidal teeth of the two mating gears are produced with the same generating circle. When mating gears of cycloidal teeth run, the common normal passes through a fixed point P (pitch point) in each position during action [Fig. 3.34]. Thus, the *law of gearing* is satisfied.

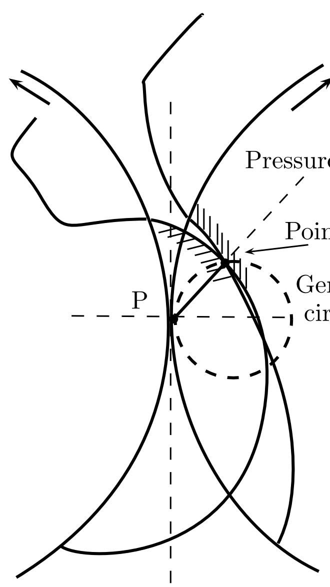


Figure 3.34 | Meshing of cycloidal teeth.

[3]Locus of a point on the circle when that circle rolls without slipping inner circumference another circle.

[4]Epicycloid is the locus of a point on the circle when that circle rolls without slipping on outer circumference of another circle.

The pressure angle varies from maximum at the beginning of engagement to zero at pitch point and then maximum but equal in reverse direction at disengagement. As is evident from Fig. 3.34, the contact between the teeth takes place along the common point between the hypocyloid and epicycloid portions of the mating teeth. Therefore, for cycloidal profile, path of approach is equal to arc of approach, and path of recess is equal to the arc of recess.

Advantage of cycloidal profile is that the phenomenon of interface does not exist. Since, these are made up of two curves, their accurate profile is difficult to produce. This has rendered this system obsolete. The details of cycloidal teeth are not discussed in the present context.

3.5.4.2 Involute Teeth

The *involute teeth*[5] consist of a single involute curve which is the locus of point on a straight (generating) line which rolls without slipping on the circumference of a base circle [Fig. 3.35].

During the messing of involute teeth of mating gears, the path of contact or pressure line is the common tangent to the base circle and it passes through the fixed point P which implies that involute teeth follow the *law of gearing* [Fig. 3.35]. When the pitch point is shifted or the center-to-center distance is increased, there is no change in velocity ratio, but there is increase in the pressure angle. Diameter of the base circle of the gear profile is a manufacturing property which remains invariant for the gear.

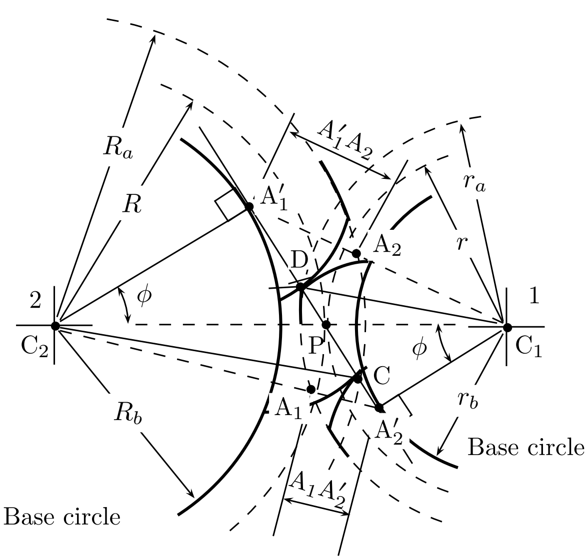


Figure 3.35 Meshing of involute teeth.

[5]An involute function is defined as

$$\text{Inv}(\phi) = \tan\phi - \phi$$

The base circle radii are related to pitch radii of the mating gears as

$$R_b = R\cos\phi$$
$$r_b = r\cos\phi$$

Initial contact occurs at point C where addendum of driven wheel intersects the line of action and final contact occurs at point D where addendum circle of the driver intersects the line of action. Between these two points, the line of action passes through the pitch point P. Using this information and trigonometry, the following properties can be determined for gears having involute teeth:

1. *Path of Contact* The path of contact consists of two components:

   (a) *Path of Approach* - The distance along the pressure line traveled by the contact point from the point of engagement (C) to the pitch point (P):

   $$\begin{aligned}\text{CP} &= \text{CA}_1' - \text{PA}_1'\\ &= \sqrt{R_a^2 - (R\cos\phi)^2} - R\sin\phi\end{aligned}$$

   The path of approach can attain a maximum value when the point of engagement lies at A_2' in Fig. 3.35. Therefore, the maximum value of path of approach is given by

   $$\text{A}_2'\text{P} = r\sin\phi$$

   (b) *Path of Recess* - The distance along the pressure line traveled by the contact point from the pitch point (P) to the point of disengagement (D):

   $$\begin{aligned}\text{PD} &= \text{DA}_2' - \text{PA}_2'\\ &= \sqrt{r_a^2 - (r\cos\phi)^2} - r\sin\phi\end{aligned}$$

   The path of recess can attain a maximum value when the point of disengagement lies at A_1' in Fig. 3.36. Therefore, the maximum value of path of recess is given by

   $$\text{PA}_1' = R\sin\phi$$

   Now, the path of contact is given by

   $$\begin{aligned}\text{CD} &= \text{CP} + \text{PD}\\ &= \sqrt{R_a^2 - (R\cos\phi)^2} - R\sin\phi\\ &\quad + \sqrt{r_a^2 - (r\cos\phi)^2} - r\sin\phi\end{aligned}$$

   The path of contact can attain a maximum value when the point of contact traces the pressure line from point E to point F. Therefore, the maximum value of path of contact is given by

   $$\text{A}_2'\text{A}_1' = (R + r)\sin\phi$$

2. Arc of Contact Using the definition of involute curve, the distance traveled by the contact point is equal to the arc on the base circle. For a gear of pitch circle radius R, the base circle radius is $R\cos\phi$. Therefore, an arc of length x on the base circle will trace same angle on the pitch circle radius, where its length will be $x/\cos\phi$. Using this, the components of arc of approach for involute teeth meshing are determined as follows:

   (a) *Arc of Approach* - Arc of approach is the distance traveled by a point on either pitch circle of the two wheels during the period of contact from engagement to pitch point:

$$\text{Arc CP} = \frac{\text{CP}}{\cos\phi}$$

   (b) *Arc of Recess* - Arc of recess is the distance traveled by a point on either pitch circle of the two wheels during the period of contact from pitch point to disengagement:

$$\text{Arc PD} = \frac{\text{PD}}{\cos\phi}$$

   Therefore, arc of contact is given by

$$\begin{aligned}\text{Arc CD} &= \text{Arc CP} + \text{Arc PD}\\ &= \frac{\text{CD}}{\cos\phi}\end{aligned}$$

3. Number of Pairs of Teeth in Contact The pair of teeth lying in between the point of engagement and point of disengagement will be meshing with each other. Therefore, the number of pairs of teeth in contact can be determined as

$$\begin{aligned}n &= \frac{\text{Arc of contact}}{\text{Circular pitch}}\\ &= \frac{\text{CDP}}{p_c\cos\phi}\end{aligned}$$

   If z_w and z_p are the number of teeth on the mating gears of equal module (m) or circular pitch ($p_c = m\pi$), then

$$\begin{aligned}R &= \frac{z_w}{2\pi}p_c\\ r &= \frac{z_p}{2\pi}p_c\end{aligned}$$

   Maximum number of teeth pairs can be in mesh when the point of contact traces the pressure line from point E to point F:

$$\begin{aligned}n_{\max} &= \frac{(R+r)\sin\phi}{p_c\cos\phi}\\ &= (z_w+z_p)\frac{\tan\phi}{2\pi}\end{aligned}$$

3.5.5 Interference

Any tooth profile can be used successfully for a spur gear as long as the mating tooth profile is compatible for producing a constant speed ratio. The compatible tooth profile is technically known as *conjugate* with respect to the tooth profile of the first gear. Mating of non-conjugate (non-involute) teeth is known as *interference*, in which the contacting teeth have different velocities which can lock the two mating gears.

A radial profile (a non-involute type) is generally adopted for the part of teeth inside the base circle. Due to inconjugate teeth, the tip of the pinion would try to dig out the flank of the tooth of the wheel. Therefore, interference occurs in the mating of two gears. Similarly, if the addendum radius of the wheel is made greater than C_2A_2' [Fig. 3.35], the tip of the wheel tooth will be in contact with a portion of the non-involute profile of the pinion tooth.

To have no interference of the teeth, the addendum should not penetrate into the base circle of the mating gear; the addendum circles of the wheel and pinion must intersect the line of action between A_2' and A_1' [Fig. 3.35]. These points are called *interference points*. To avoid interference, the limiting value of addendum of the wheel is A_1A_2' whereas that of pinion is $A_1'A_2$ and the latter is clearly greater than the former ($A_1'A_2 > A_1A_2'$). Therefore, for equal addenda[6] of the wheel and the pinion, the addendum radius of the wheel decides whether the interference will occur or not.

The addendum of wheel can be determined geometrically as follows:

$$\begin{aligned}C_2A_2'^2 &= C_2A_1'^2 + A_1'A_2'^2\\ &= C_2A_1'^2 + \left(A_1'P + PA_2'\right)^2\\ &= (R\cos\phi)^2 + (R\sin\phi + r\sin\phi)^2\\ &= R^2 + \left(r^2+2rR\right)\sin^2\phi\\ &= R^2\left[1+\frac{1}{R^2}\left(r^2+2rR\right)\sin^2\phi\right]\\ &= R^2\left[1+\left(\frac{r^2}{R^2}+\frac{2r}{R}\right)\sin^2\phi\right]\end{aligned}$$

Therefore,

$$C_2A_2' = R\sqrt{1+\frac{r}{R}\left(\frac{r}{R}+2\right)\sin^2\phi}$$

[6] The radial distance between the pitch circle and the addendum circle. Addenda is plural form of word 'addendum'.

Maximum value of the addendum of the wheel is

$$\begin{aligned} A_1A_2' &= C_2A_2' - R \\ &= R\sqrt{1+\frac{r}{R}\left(\frac{r}{R}+2\right)\sin^2\phi} - R \\ &= R\left[\sqrt{1+\frac{r}{R}\left(\frac{r}{R}+2\right)\sin^2\phi} - 1\right] \end{aligned}$$

If the selected value of addendum of larger gear (wheel) is a_w times of module m, then

$$A_1A_2' \geq a_w m$$

$$R\left[\sqrt{1+\frac{r}{R}\left(\frac{r}{R}+2\right)\sin^2\phi} - 1\right] \geq a_w m$$

Let T be the number of teeth on wheel, then

$$m = \frac{2R}{T}$$

Gear ratio is expressed as

$$\frac{R}{r} = G$$

From the above relations,

$$T \geq \frac{2a_w}{\sqrt{1+(1/G)(1/G+2)\sin^2\phi} - 1} \tag{3.26}$$

When addendum is equal to module ($a_w = 1$),

$$T \geq \frac{2}{\sqrt{1+(1/G)(1/G+2)\sin^2\phi} - 1} \tag{3.27}$$

This is the expression for number of teeth on a larger gear (wheel) to avoid interference in meshing of gears at pressure angle ϕ, gear ratio G, and addendum equal to module. A gear-meshing having pressure angle $\phi = 20°$ and unit gear ratio ($G = 1$) will be interference-free if $T > 12.32 \approx 13$.

3.5.6 Rack and Pinion

In rack and pinion arrangement, the pinion rotates while the rack translates. This mechanism is used to convert either rotary motion into linear motion or vice versa. In this mechanism, the spur rack can be considered to be a spur gear of infinite pitch radius with its axis of rotation placed at infinity parallel to that of the pinion; instead of base circle, the larger gear is formed over a straight line.

The condition for interference running of a rack-pinion arrangement can also be examined, as depicted in Fig. 3.36. To avoid interference, the addendum of pinion should not penetrate to the pitch line (PA_1) of rack. Let z_p be the number teeth on pinion and ϕ be the pressure angle. The depth A_1A_2' can be determined as follows:

$$\begin{aligned} A_1A_2' &= (r\sin\phi)\sin\phi \\ &= r\sin^2\phi \\ &= \frac{mz_p}{2}\sin^2\phi \end{aligned}$$

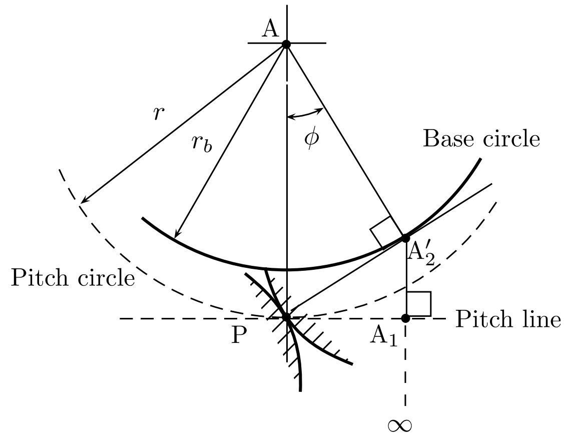


Figure 3.36 | Interference in rack-pinion.

For addendum equal to module, the running will be free of interference if

$$\begin{aligned} A_1A_2' &\geq m \\ \frac{mz_p}{2}\sin^2\phi &\geq m \\ z_p &\geq \frac{2}{\sin^2\phi} \end{aligned}$$

For interference-free running of a rack pinion having $\phi = 20°$, $z_p > 18$, and for $\phi = 14.5°$, $z_p > 32$.

3.5.7 Helical Gears

On the basis of the direction of helix (ψ) cut on *helical gears*, these can be either right handed or left handed. Angle between axes of two shafts is given by

$$\theta = \psi_1 - \psi_2 \tag{3.28}$$

The type of contact between teeth of helical gear teeth depends on θ. For parallel shafts ($\theta = 0$), the contact exists along a diagonal line, while for skew shafts ($\theta \neq 0$), there exists a point contact between gear teeth. Therefore, crossed-helical or spiral gears are not used for heavy loads, but parallel helical gears are stronger than spur gears because of diagonal contacts. When helical gears are used in skew shafts, they are called spiral gear or crossed-helical gears.

In the case of helical gears, normal circular pitch of two mating gears must be the same. If the helix angle of a helical gear is increased, the load carrying capacity of the tooth increases and the form factor increases with increase in helix angle.

3.5.8 Gear Trains

Desired speed ratio in a gear system can be achieved by combination of various gears with different number of teeth and inversions. These combinations are called *gear trains*. Gear trains are necessary when a large or certain velocity ratio or mechanical advantage is required, or shafts are kept at a distance. Some important types of gear trains are discussed in following sections.

3.5.8.1 Simple Gear Train When each gear of the gear train is mounted on a separate shaft and all the gear axes remain in position fixed to frame, it is called *simple gear train*.

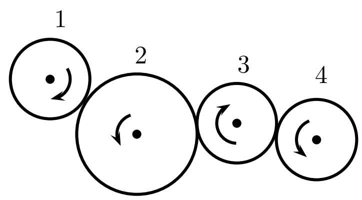


Figure 3.37 | Simple gear train.

For the simple gear train shown in Fig. 3.37, the speed ratio, also called *train value*, is given by

$$N_4 = N_1 \times \frac{T_2}{T_1} \times \frac{T_3}{T_2} \times \frac{T_4}{T_3}$$

$$\frac{N_4}{N_1} = \frac{T_4}{T_1}$$

Thus, in simple gear trains, the intermediate gears do not have any influence on the velocity ratio. Intermediate gears work as *idler gears*, serving two purposes: first, they control the direction of rotation of output gear, and second, they bridge the gap between the shafts.

3.5.8.2 Compound Gear Train A gear pair is called compounded if they are mounted on the same shaft and are made into an integral part in some way. A *compound gear train* consists of one or more compound gear pairs. The compounding involves large speed reduction.

Compound gear trains are of two classes: reverted and non-reverted. In *reverted gear trains*, the first and last gears are co-axial [Fig. 3.38], otherwise it is called *non-reverted gear train* [Fig. 3.39]. Reverted gear trains are used in lathe machines where back gear is used to impart slow speed of the chuck.

3.5.8.3 Epicyclic Gear Train A gear train having a relative motion of axes is called *planetary* or *epicyclic gear train*. The axis of at least one of the gear also moves relative to the frame. For example, in Fig. 3.40, if the gear S is made fixed, the axis of gear P will rotate about the axis of S. Such a gear train has two degrees of freedom. Planetary gear trains are quite useful in making the reduction unit more compact than a compound gear train.

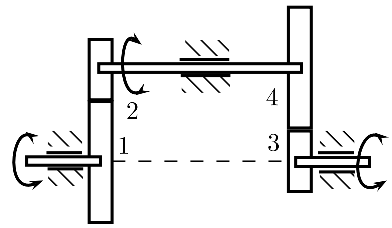


Figure 3.38 | Compound gear train (reverted).

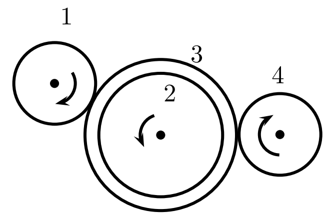


Figure 3.39 | Compound gear train (non-reverted)

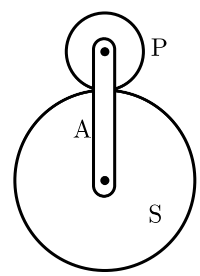


Figure 3.40 | Epicyclic gear train.

To simplify the analysis, an epicyclic gear train can be converted into a simple reverted gear train by fixing the arm and releasing the fixed gear. The relative motion between any two links does not change but the absolute motion does change. Based on this understanding, the following procedure is adopted in analyzing the epicyclic gear trains:

1. Lock arm to obtain simple reverted train such that except arm, all gears are free to rotate.
2. Consider any gear and turn it in say clockwise direction through one revolution. Establish corresponding revolutions of all other gears, note it in a tabular form in a first row.

3. Rotate the chosen gear in step 2 x times, by multiplying each entry of the first row by an arbitrary variable x and enter the values of the product in the second row of the table.

4. Revolve arm by y revolutions by adding y to all the entries of second row, and enter the results in third row.

In the above steps, x and y are the two unknown variables in terms of which the third row gives corresponding revolutions of each gear and arm. Values of x and y are determined by the given condition, for example, sun is fixed on frame, so its speed is zero, and so on. The revolutions of all the gears are determined by putting values of x and y.

To better understand the procedure, consider an epicylic gear train shown in Fig. 3.41. The gear train consists of a sun wheel S, a stationary internal gear E and three identical planet wheels P carried on a star-shaped carrier C. The size of different wheels are such that the planet carrier C rotates at 1/4th of the speed of the sun wheel S. The minimum number of teeth on any wheel is 12. The problem is to determine the number of teeth on sun gear.

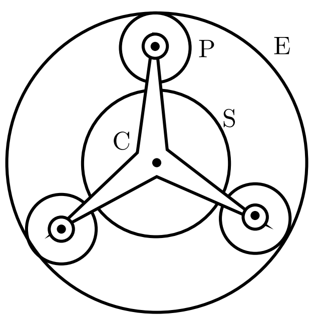


Figure 3.41 Epicyclic gear train.

The tabular method is shown in Table 3.1.

Table 3.1 Tabular method for epicyclic gear train

Condition	C	S	P	E
C fixed, S one revolution	0	1	$-\dfrac{T_S}{T_P}$	$-\dfrac{T_S}{T_E}$
C fixed, S x revolutions	0	x	$-\dfrac{T_S}{T_P}x$	$-\dfrac{T_S}{T_E}x$
Add y to all	y	$y+x$	$y-\dfrac{T_S}{T_P}x$	$y-\dfrac{T_S}{T_E}x$

Given that $N_C = N_S/4$. Therefore, using the speeds of gears shown in the bottom row,

$$N_C = \frac{N_S}{4}$$
$$y = \frac{y+x}{4}$$
$$\frac{x}{y} = 3$$

Also,

$$N_E = 0$$
$$y - \frac{T_S}{T_E}x = 0$$
$$T_E = \frac{x}{y}T_S$$
$$T_E = 3T_S$$

Taking the dimensional constraints,

$$T_S + 2T_P = T_E$$
$$T_P = \frac{T_E - T_S}{2}$$
$$T_P = T_S$$

Observing above equations,

$$T_E > T_S$$

Thus, the minimum number of teeth is $T_S = 12$ (given). Using above relations,

$$T_S = 12$$
$$T_E = 36$$
$$T_P = 12$$

3.6 DYNAMIC ANALYSIS OF SLIDER CRANK MECHANISM

Figure 3.42 shows the slider crank mechanism under consideration where l and r are the lengths of connecting rod and crank, respectively. The crank has angular speed ω and angular acceleration α.

Ratio of lengths l and r is denoted as

$$n = \frac{l}{r}$$

The relation between angular position of connecting rod (θ) and crank (ϕ) is determined as

$$l \sin\phi = r \sin\theta$$
$$\sin\phi = \frac{\sin\theta}{n} \tag{3.29}$$

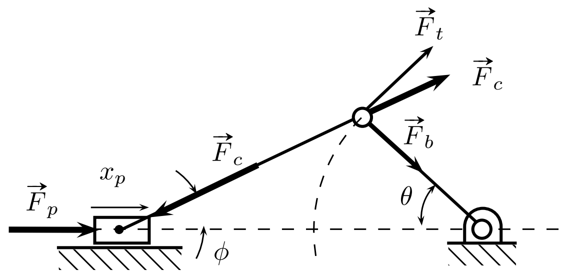


Figure 3.42 Slider-crank mechanism.

3.6.1 Motion in Links

3.6.1.1 Slider The distance of slider at extreme outer position is $(l+r)$, therefore, the position or displacement function of slider, measured from the extreme outer position, at any instant is given by

$$\begin{aligned} x_p &= (l+r) - l\cos\phi - r\cos\theta \\ &= r\left\{(1-\cos\theta) + \frac{\sin^2\theta}{2n}\right\} \end{aligned}$$

Therefore, velocity of the slider, directed towards crank pin, at any instant is determined as

$$\begin{aligned} v_p &= \frac{dx_p}{dt} \\ &= r\omega\left\{\sin\theta + \frac{\sin 2\theta}{2n}\right\} \end{aligned}$$

Similarly, acceleration of the slider is given by

$$\begin{aligned} f_p &= \frac{dv_p}{dt} \\ &= r\omega^2\left\{\cos\theta + \frac{\cos 2\theta}{n}\right\} \end{aligned}$$

3.6.1.2 Connecting Rod The angle ϕ denotes the angular position of connecting rod with respect to frame. Therefore, angular speed of connecting rod is determined using Eq. (3.29):

$$\begin{aligned} \cos\phi\frac{\partial\phi}{\partial t} &= \omega\cos\theta \\ \omega_{\text{CR}} &= \frac{\partial\phi}{\partial t} \\ &= \frac{\omega}{n}\times\frac{\cos\theta}{\cos\phi} \end{aligned}$$

For small values of ϕ ($\cos\phi \approx 1$):

$$\omega_{\text{CR}} = \omega\frac{\cos\theta}{n}$$

Angular acceleration of connecting rod is derived as

$$\begin{aligned} \alpha_{\text{CR}} &= \frac{d\omega_{\text{CR}}}{dt} \\ &= -\omega^2\frac{\sin\theta}{n} \end{aligned}$$

For special case $\theta = \phi = 0$:

$$\begin{aligned} (\omega_{\text{CR}})_{\max} &= \frac{\omega}{n} \\ \alpha_{\text{CR}} &= 0 \end{aligned}$$

3.6.2 Dynamic Forces

3.6.2.1 Piston Effort The net effective force acting on the piston is known as *piston effort*. It consists of three elements:

1. *Pressure Force*

$$F_p = p_1A_1 - p_2A_2$$

2. *Inertia Force*

$$F_i = m\omega^2 r\left[\cos\theta + \frac{\cos 2\theta}{n}\right]$$

3. *Gravity Force*

$$F_g = mg$$

The piston effort is given by

$$F_P = F_p - F_i \pm F_g$$

Gravity force is considered only in vertical motion of the piston with suitable sign.

3.6.2.2 Crank Effort The force acting on the crank at the joint of connecting rod as a result of the force on the piston is known as *crank effort*. Resolving the piston force along the connecting rod,

$$F_{\text{CR}} = \frac{F_P}{\cos\phi}$$

The force exerts a thrust on cylinder-wall, given by

$$F_{\text{wall}} = F_{\text{CR}}\sin\phi$$

The force on the crank generates crank effort, given by

$$F_t = F_{\text{CR}}\sin(\phi+\alpha)$$

The force transmitted to the crank bearing is

$$F_B = F_{\text{CR}}\cos(\phi+\alpha)$$

3.6.3 Turning Moment

Turning moment on the crank shaft is derived as

$$\begin{aligned}T &= F_t \times r \\ &= \frac{F_P}{\cos\phi} \times \sin(\theta+\alpha) \times r \\ &= \frac{rF_P}{\cos\phi}\{\sin\theta\cos\phi + \cos\theta\sin\phi\} \\ &= F_P r\left\{\sin\theta + \cos\theta\sin\phi\left(\frac{1}{\cos\phi}\right)\right\} \\ &= F_P r\left\{\sin\theta + \frac{\sin 2\theta}{2\sqrt{n^2-\sin^2\theta}}\right\}\end{aligned}$$

This is the expression of turning moment in terms of piston effort (F_p), crank radius r for given angular position (θ) of the crank. For given values of F_p and r, the turning moment is can be represented as

$$T = f(\theta)$$

Using this function, $T(\theta)$ can be plotted for different values of θ. This plot is called *turning moment diagram.*

3.7 FLYWHEEL

In every machine, there is at least one point at which energy is supplied and at least one other point at which energy is delivered. Between these two points, there is undesired variation in energy and speed of the machine. A *flywheel* is a balanced spinning wheel with significant moment of inertia. It is used to control variation in speed during each cycle of an engine by making moment of inertia of rotating parts quite large. A flywheel acts as a reservoir of energy which stores and releases the energy as per its requirements.

3.7.1 Mean Speed of Rotation

Let a machine be attached with a flywheel with moment of inertia[7] I. The maximum and minimum seeds of rotation of the flywheel are ω_1 and ω_2, respectively. Therefore, the mean speed of rotation of the flywheel is

$$\omega = \frac{\omega_1+\omega_2}{2}$$

To quantify the fluctuation of speed, a term *coefficient of fluctuation of speed* (k_s) is defined as

$$k_s = \frac{\omega_1-\omega_2}{\omega}$$

[7]Moment of Inertia of a disc is

$$I_{\text{disc}} = \int_0^R \rho \times 2\pi b r dr \times r^2 = 2\frac{\rho b\pi R^2 \times R^2}{4} = \frac{mR^2}{2}$$

Using the definition of mean speed ω, and taking $r = \omega_1/\omega_2$:

$$\begin{aligned}\frac{k_s}{2} &= \frac{r-1}{r+1} \\ rk_s + k_s &= 2r-2 \\ r(2-k_s) &= 2+k_s \\ r &= \frac{2+k_s}{2-k_s}\end{aligned}$$

This expression relates speed ratio r to the *coefficient of fluctuation of speed* (k_s).

3.7.2 Energy Fluctuation

Let E be the kinetic energy of the flywheel at mean speed ω:

$$E = \frac{1}{2}I\omega^2$$

The maximum fluctuation of energy (e_{max}) is determined as

$$\begin{aligned}e_{\text{max}} &= \frac{1}{2}I(\omega_1^2-\omega_2)^2 \\ &= I(\omega_1-\omega_2)\frac{\omega_1+\omega_2}{2} \\ &= I\frac{(\omega_1-\omega_2)}{\omega}.\omega^2 \\ &= I\omega^2 k_s \\ &= 2\times\frac{I\omega^2}{2}k_s \\ &= 2k_s E\end{aligned}$$

Thus,

$$k_s = \frac{e_{\text{max}}}{2E}$$

A flywheel stores the extra energy (more than average) supplied by engine during power stroke, and to reduce variations in the power generation curve, it returns the stored energy. Therefore, a flywheel must have sufficient moment of inertia to become capable of absorbing the maximum energy variation.

3.7.3 Turning Moment Diagrams

A *turning moment diagram* indicates the variation in the turning moment or torque due to the pressure variation in the cylinder for one complete revolution of the power cycle. The profile and frequency of *turning moment diagram* depends upon the type of engine or power pack being in use.

Let torque equation (T–θ) on turning diagram be presented as

$$T = f(n\theta)$$

Thus, the cycle repeats in every $2\pi/n$ radian. The mean torque of the cycle will be given by

$$\bar{T} = \frac{n}{2\pi}\int_0^{2\pi/n} T d\theta$$

The excess energy is calculated by first finding θ_A and θ_B between which maximum fluctuation (addition or delivery) of energy occurs above the mean value. For this, the following equation is solved for N different roots of θ:

$$T - \bar{T} = 0$$

The variation in energy is calculated between the successive roots of θ using the following equation (n varying from 1 to N):

$$e = \int_{\theta_n}^{\theta_{n+1}} (T - \bar{T}) d\theta$$

Out of such variations, θ_A and θ_B are chosen between which the maximum variation of energy is given by

$$e_{\max} = \int_{\theta_A}^{\theta_B} (T - \bar{T}) d\theta$$

Maximum and minimum speed occurs at the points of maximum and minimum kinetic energy.

3.8 BELT DRIVE

A belt is a looped strip of flexible material used to mechanically link two or more rotating shafts. A belt drive offers smooth transmission of power between two shafts at considerable distance. The drive is lubrication-free and requires minimum maintenance. The pulleys are usually less expensive than chain drive sprockets and exhibit little wear over long periods of operation.

The *belt drive* is not a positive drive. Let s_1 and s_2 be the slips in driver pulley and driven pulley, respectively. Let N_1 and N_2 denote speeds of driver and driven pulleys having diameter D and d, respectively. For belt thickness t, speed ratio of the drive is given by

$$\frac{N_2}{N_1} = (1 - s_1)(1 - s_2)\frac{D + t}{d + t}$$

Therefore, the total slip is

$$s = (1 - s_1)(1 - s_2)$$

Belt drives have drawback of slippage, wear in belts, and unsuitability in adverse service conditions. Slip in *timing belts* (used in time crank and cam) is zero because they are toothed belts.

3.8.1 Types of Belt Drives

In a two-pulley system, the belt can either drive the pulleys in the same direction (open drive), or the belt can be crossed so that the direction of the shafts is opposite (crossed drive) [Fig. 3.43].

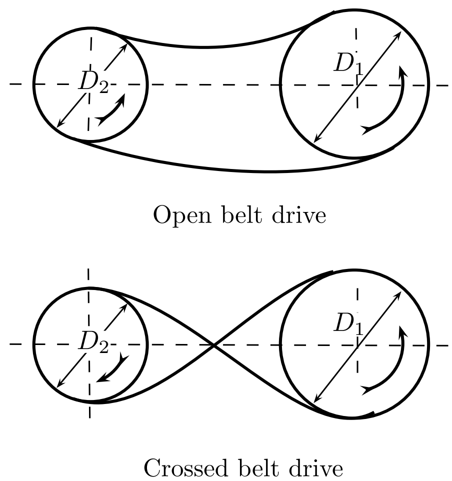


Figure 3.43 Open and crossed belt drives.

The two types of belt drives are explained as follows:

1. Open Belt Drive An *open belt drive* is used to rotate the driven pulley in the direction of driving pulley. Power transmission results makes one side of the belt more tightened (*tight side*) as compared to the other (*slack side*). In horizontal drives, tight side is always kept in the lower side of two pulleys because the sag of the upper side slightly increases the angle of wrap of the belt on the two pulleys.

2. Crossed Belt Drive A *crossed belt drive* is used to rotate driven pulley in the opposite direction of the driving pulley. Higher the value of wrap enables more power can be transmitted than an open belt drive. However, bending and wear of the belt are important concerns.

3.8.2 Length of Belts

The length of the belt depends upon the type of belt drive. The expressions are derived as follows:

1. Open Belt Drive Consider an open belt drive consisting of two pulleys of diameters D and d kept apart by a center distance c [Fig. 3.44].

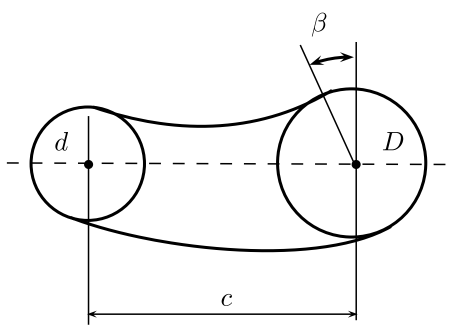


Figure 3.44 | Open belt drive.

Angle of lap β is related as

$$\sin\beta \approx \beta = \frac{D-d}{2c}$$
$$\cos\beta = \sqrt{1-\sin^2\beta} = \left\{1-\left(\frac{D-d}{2c}\right)^2\right\}^{1/2} \approx 1-\frac{1}{2}\frac{(D-d)^2}{4c^2}$$

These expressions for β and $\cos\beta$ is used as

$$\begin{aligned} L &= 2\left\{\frac{\pi}{2}-\beta\right\}\frac{d}{2} \\ &+2\left\{\frac{\pi}{2}+\beta\right\}\frac{D}{2} \\ &+2c\left\{1-\frac{1}{2}\frac{(D-d)^2}{4c^2}\right\} \\ &= 2\left\{\frac{\pi}{2}-\frac{D-d}{2c}\right\}\frac{d}{2} \\ &+2\left\{\frac{\pi}{2}+\frac{D-d}{2c}\right\}\frac{D}{2} \\ &+2c\left\{1-\frac{1}{2}\frac{(D-d)^2}{4c^2}\right\} \\ &= \pi\frac{(D+d)}{2}+2c+\frac{(D-d)^2}{4c} \end{aligned}$$

2. *Crossed Belt Drive* Consider a crossed belt drive consisting of two pulleys of diameters D and d kept apart by a distance c [Fig. 3.45].

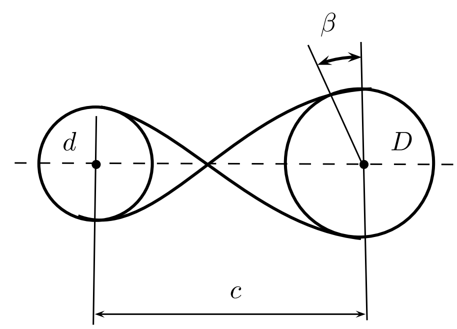


Figure 3.45 | Crossed belt drive.

Angle of lap β is related as

$$\sin\beta \approx \beta = \frac{D+d}{2c}$$
$$\cos\beta = \sqrt{1-\sin^2\beta} = \left\{1-\left(\frac{D+d}{2c}\right)^2\right\}^{1/2} \approx 1-\frac{1}{2}\frac{(D+d)^2}{4c^2}$$

Length of the belt is expressed as

$$\begin{aligned} L &= 2\left(\frac{\pi}{2}+\beta\right)\frac{d}{2}+2\left(\frac{\pi}{2}-\beta\right)\frac{D}{2}+2c\cos\beta \\ &= 2\left\{\frac{\pi}{2}+\frac{D+d}{2c}\right\}\frac{d}{2}+2\left\{\frac{\pi}{2}-\frac{D+d}{2c}\right\}\frac{D}{2} \\ &+2c\left\{1-\frac{1}{2}\frac{(D+d)^2}{4c^2}\right\} \\ &= \pi\frac{(D+d)}{2}+2c+\frac{(D+d)^2}{4c} \end{aligned}$$

3.8.3 Power Transmission

Let a belt of mass m per unit length rotate with peripheral velocity v over a pulley of radius r [Fig. 3.46]. An elemental length $\delta l = r\delta\theta$ of the belt passing over the pulley is subjected to centrifugal force, which creates centrifugal tension T_c both on tight and slack sides. Let μ be the coefficients of friction and R be the reaction between belt and pulley.

3.8.3.1 Initial Tension The belt is assembled with an initial tension. When power is transmitted, the tension in tight side increases from T_i to T_1 and on the slack side decreases from T_i to T_2. If the belt is assembled to obey Hooke's law and its length remains constant, then

$$T_1 - T_i = T_i - T_2$$
$$T_i = \frac{T_1+T_2}{2}$$

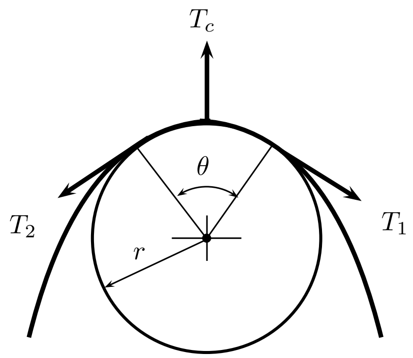


Figure 3.46 Centrifugal tension in belt drives.

3.8.3.2 Centrifugal Tension The centrifugal tension in the belt is determined as

$$m\,(r d\theta)\,\frac{v^2}{2} = 2T_c\frac{d\theta}{2}$$
$$T_c = mv^2$$

3.8.3.3 Ratio of Driving Tensions Equilibrium of the belt element is examined in two orthogonal directions:

1. *Radial Direction*

$$R + T_c d\theta = (T + dT)\,\frac{d\theta}{2} + T\frac{d\theta}{2}$$
$$R = (T - T_c)\,d\theta \qquad (3.30)$$

2. *Tangential Direction*

$$T + dT - T = \mu R$$
$$dT = \mu R \qquad (3.31)$$

Using Eqs. (3.30) and (3.31),

$$dT = \mu\,(T - T_c)\,d\theta$$
$$\int_{T_1}^{T_2} \frac{dT}{T - T_c} = \int_0^{\theta} \mu d\theta$$
$$\frac{T_1 - T_c}{T_2 - T_c} = e^{\mu\theta}$$

where T_1 and T_2 are actual tensions in the belt, $T_1 - T_c$ and $T_2 - T_c$ are the driving tension or effective tensions on the pulley. The equation indicates that if the diameter of the driving and driven pulleys are unequal, the belt will slip first on the pulley having smaller angle of lap θ (i.e. smaller pulley in open belt, friction will be less). Therefore, the *angle of lap* (also called *angle of wrap* or *contact*) is taken for the angle subtended by the segment of belt in contact with smaller pulley at the center.

The idler pulleys are used to tighten the belts and increases the angle of lap. Idler pulleys are held against the belt by their own weight in addition to an adjustable weight. Idler pulley are not provided crowning, and are kept on loose sides, nearer to smaller pulley.

3.8.3.4 Power Transmitted The power transmitted through the belt drive is determined as

$$\begin{aligned} P &= (T_1 - T_2)\,v \\ &= \{(T_1 - T_c) - (T_2 - T_c)\}\,v \\ &= (T_1 - T_c)\left\{1 - e^{-\mu\theta}\right\} v \\ &= \left(T_1 - mv^2\right)\left\{1 - e^{-\mu\theta}\right\} v \end{aligned}$$

For maximum power transmission,

$$\frac{dP}{dv} = 0$$
$$\frac{d}{dv}\left(T_1 - mv^2\right) v = 0$$
$$T_1 - 3mv^2 = 0$$
$$T_c = \frac{T_1}{3}$$

Belt-velocity for maximum power transmission is

$$v^* = \sqrt{\frac{T_1}{3m}}$$

At this velocity, the maximum possible power transmission is

$$\begin{aligned} P &= \left(T_1 - \frac{T_1}{3}\right)\left\{1 - e^{-\mu\theta}\right\} v \\ &= \frac{2T_1}{3}\left\{1 - e^{-\mu\theta}\right\} v \end{aligned}$$

3.8.3.5 Effect of Centrifugal Tension If centrifugal tension is neglected, then

1. *Ratio of Driving Tensions*

$$\frac{T_1}{T_2} = e^{\mu\theta}$$

2. *Power Transmitted*

$$P = T_1\left\{1 - e^{-\mu\theta}\right\} v$$

Therefore, the effect of centrifugal is to reduce the power transmitted. The maximum efficiency of belt drive remains unaffected to 66.67%.

3.8.4 Crowning of Pulleys

Pulleys of flat drives are crowned by producing slightly conical or convex surface on the rim. When the belt slips off the pulley, the crown helps it to adhere to the cone surface due to pull on the belt. Crowning is always done on the driving pulley because tension is on entry side.

3.8.5 Law of Belting

According to *law of belting*, the center line of the belt when it approaches a pulley must lie in the mid-plane of the pulley. Hence, in non-parallel shafts, rotation is possible in only one direction.

3.8.6 Elastic Creep

Presence of friction between pulley and belt causes differential tension in the belt. This differential tension causes the belt to elongate or contract, and thus create a relative motion between the belt and the pulley surface. This slip is called *elastic creep*. This reduces the speed of the belt and power transmission.

3.8.7 V-Belts

In V-belt drives, the angle of groove β results in a higher coefficient of friction, known as *effective coefficient of friction*, given by

$$\mu' = \frac{\mu}{\sin(\beta/2)} \tag{3.32}$$

which is always greater than μ. This is the main reason behind use of V-belts. V-belt groove angle is 34—36° while pulley groove angle is 40°. To avoid locking, V-belts are designed not to touch the bottom of the V-groove.

3.9 FRICTION

The sliding of one solid body in contact with a second solid body is always restricted by a force called the force of *friction*. It acts in opposite direction to that of the relative motion and is tangential to the surface of the two bodies at the point of contact.

Friction is an important aspect in every machine because it involves wearing of machine components and consumes energy that dissipates into heat. Sometimes friction is also desirable for functioning of a machine, such as friction clutches, belt drives.

3.9.1 Theory of Friction

Consider a body of weight W resting on a smooth and dry plane surface. The normal reaction at the surface is R_n [Fig. 3.47].

If a small horizontal force F is applied to the body to move it over the surface, until the body is unable to move, the equilibrium equation of the body is given by

$$R_n = W$$
$$F = F'$$

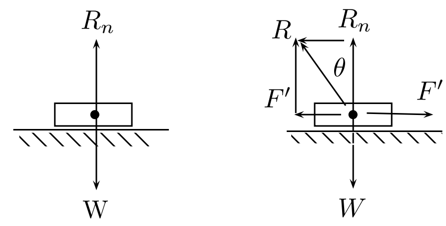


Figure 3.47 Coefficient of friction.

where F' is the horizontal force resting on the motion of the body. Let R be the resultant of R_n and F', expressed as

$$R_n = R\cos\theta$$
$$F' = R\sin\theta$$

If the pull F is increased continuously, the above angle θ would attain a limiting value ϕ [Fig. 3.48] when the body will just move and with μ as the *coefficient of friction*,

$$F' = \mu R_n$$
$$\tan\phi = \frac{F'}{R_n} = \mu$$

Here, ϕ is called *angle of friction*.

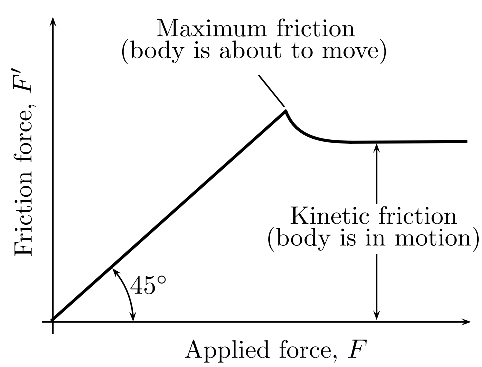


Figure 3.48 Friction force versus applied force.

The cone of friction is the imaginary cone generated in the case of non-coplanar forces by revolving the static resultant reaction R about the normal. Its cone angle will be 2ϕ [Fig. 3.49].

3.9.1.1 Friction Circle Greasy friction occurs in heavy loaded, slow running bearings. When a shaft rests in its bearing, the weight of the shaft W acts through its center of gravity. The reaction of the bearing acts in line with W in the vertically upward direction. The shaft rests on the bearing in metal-to-metal contact.

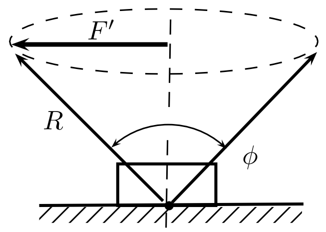


Figure 3.49 | Cone of friction.

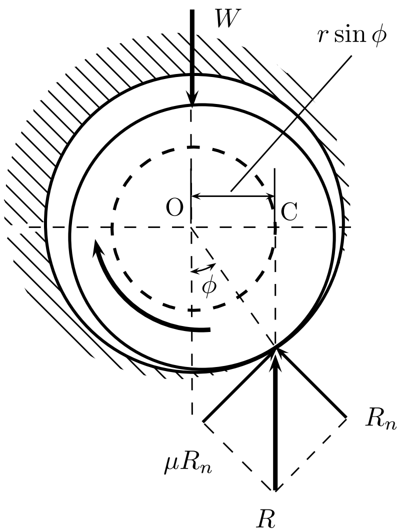


Figure 3.50 | Friction circle.

When a torque is applied to the shaft, it rotates and seat of pressure creeps or climbs up the bearing in a direction opposite to that of rotation. Metal-to-metal contact still exists and greasy friction criterion applies as the oil film will be of molecular thickness. For equilibrium, the resultant reaction R must vertically act upward and must be equal to W, however, these two forces at a distance OC, constitute a couple [Fig. 3.50]. If r is the radius of journal (shaft), the friction torque will be given by

$$T = W \times r \sin\phi$$

A circle drawn with radius $r\sin\phi$ is known as the *friction circle* of the journal.

3.9.2 Inclined Plane

Consider a body of weight W over an inclined plane at an angle α to the horizontal. The limiting angle of friction between the surfaces is ϕ. Let a force F be applied to cause the body slide with uniform velocity parallel to the slope. On the limiting case, it will be equal to the angle of friction ϕ.

Three situations are possible for body over inclined plane: at rest, moving up, and moving down the plane. The expressions for forces and efficiency in such cases are derived as follows.

3.9.2.1 Body at Rest For the equilibrium of the body at rest on the plane [Fig. 3.51], the limiting resultant friction force R is given by

$$\begin{aligned} W\sin\alpha &= \mu W\cos\alpha \\ \tan\alpha &= \mu \\ &= \tan\phi \\ \alpha &= \phi \end{aligned}$$

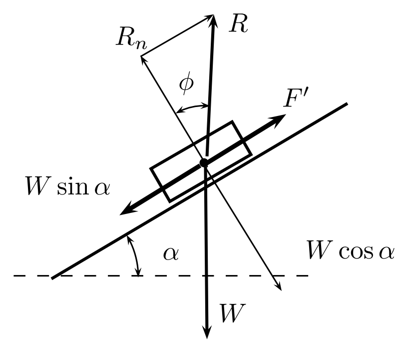


Figure 3.51 | Body at rest on the plane.

Angle of repose is the maximum angle to which an inclined plane can be raised before an object resting on it starts moving under the action of its won weight and friction resistance. Above equation shows that angle of response is equal to the angle of friction.

3.9.2.2 Body Moving Up the Plane For the equilibrium of the body moving up the plane [Fig. 3.52], using *Lami's theorem*:

$$\frac{R}{\sin\theta} = \frac{F}{\sin(\alpha+\phi)} = \frac{W}{\sin\{\theta-(\alpha+\phi)\}}$$

Therefore

$$\frac{F}{W} = \frac{\sin(\phi+\alpha)}{\sin\{\theta-(\alpha+\phi)\}} \tag{3.33}$$

In the above equation, the pulling force will be minimum if the denominator on right hand side is maximum, therefore,

$$\begin{aligned} \sin\{\theta-(\alpha+\phi)\} &= 1 \\ \theta-(\alpha+\phi) &= \frac{\pi}{2} \\ \theta-\left(\frac{\pi}{2}+\alpha\right) &= \phi \end{aligned}$$

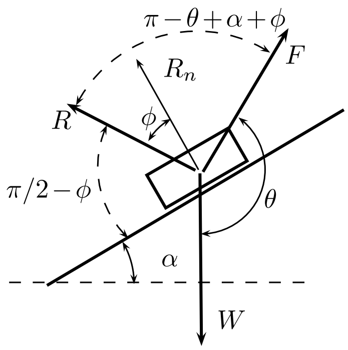


Figure 3.52 | Motion up the plane.

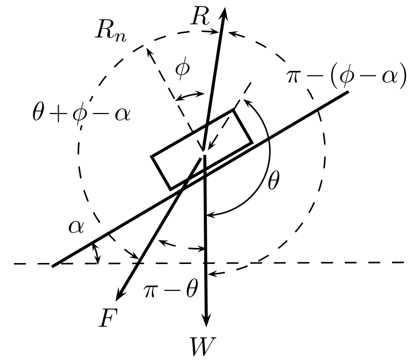


Figure 3.53 | Motion down the plane.

This equation indicates that the pulling force (F) will be minimum if the angle between F and inclined plane is equal to the angle of friction. In that case, the minimum pulling force will be given by

$$F = W\sin(\alpha+\phi)$$

In this case, *efficiency* of inclined plane is the ratio of the pulling force required to move the body without friction ($\mu = 0$) and with friction. Therefore, using Eq. (3.33),

$$\begin{aligned}\eta &= \frac{F_0}{F}\\ &= \frac{\sin\alpha}{\sin(\theta-\alpha)} \times \frac{\sin\{\theta-(\alpha+\phi)\}}{\sin(\alpha+\phi)}\\ &= \frac{\cot(\phi+\alpha)-\cot\theta}{\cot\alpha-\cot\theta}\end{aligned}$$

In the above equation, when pulling force F is applied in horizontal direction ($\theta = \pi/2$),

$$\eta = \frac{\tan\alpha}{\tan(\alpha+\phi)}$$

3.9.2.3 Body Moving Down the Plane For the equilibrium of the body moving down the plane [Fig. 3.53], using *Lami's theorem*:

$$\frac{F}{\sin\{\pi-(\phi-\alpha)\}} = \frac{W}{\sin\{\theta-\alpha+\phi\}}$$

$$\frac{F}{W} = \frac{\sin(\phi-\alpha)}{\sin\{\theta+(\phi-\alpha)\}} \tag{3.34}$$

If friction is neglected in the above case, then

$$F = -\frac{-W\sin\alpha}{\sin(\theta-\alpha)}$$

Negative sign in the above expression indicates that the weight component adds the effort to move the body in the downward direction, therefore, force is required in opposite direction to oppose the downward motion.

In this case, *efficiency of inclined plane* is the ratio of the force required to move the body with friction and without friction ($\mu = 0$). Therefore, using Eq. (3.34),

$$\begin{aligned}\eta &= \frac{F}{F_0}\\ &= \frac{\sin(\phi-\alpha)}{\sin\{\theta+(\phi-\alpha)\}} \times \frac{\sin(\theta-\alpha)}{-\sin\alpha}\\ &= \frac{\cot\alpha-\cot\theta}{\cot(\phi-\alpha)+\cot\theta}\end{aligned}$$

In the above equation, when pulling force F is applied in horizontal direction ($\theta = \pi/2$),

$$\eta = \frac{\tan(\phi-\alpha)}{\tan\alpha}$$

3.9.3 Friction in Screw Threads

Screw and nut combinations are used to convert rotary motion into translational motion and transmit power. Screw threads are mainly of two types namely, square threads and V-threads. The V-threads offer more resistance to the motion than square threads, therefore, the square threads are used in screw jacks, whereas V-threads are used for tightening of two components.

The axial distance traveled by thread in one turn is called lead. Pitch (p) is the distance between two adjacent threads parallel to axis of the screw. For single start threads, lead is equal to pitch. *Helix angle* (α) is the slope or inclination of the threads with horizontal. For single start screws [Fig. 3.54], if d is the mean diameter of the helix, then

$$\tan\alpha = \frac{p}{\pi d}$$

In the V-threads shown in Fig. 3.55, the reaction on thread is inclined by angle β with vertical direction of load W. Therefore, due to wedge effect, the *effective*

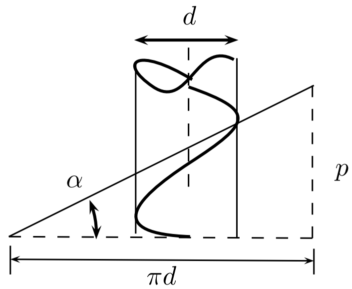


Figure 3.54 | Helix angle of screw threads.

coefficient of friction is given by

$$\mu' = \frac{\mu}{\cos(\beta/2)}$$

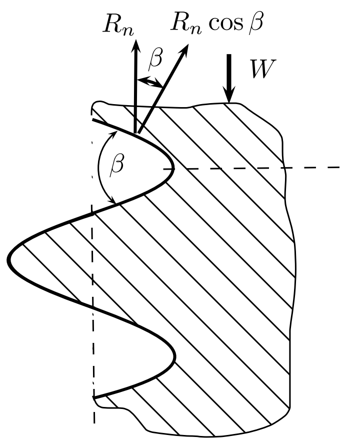


Figure 3.55 | Friction in V-threads.

The following derivations are for square threads but V-threads can be analyzed by converting these derivations into equivalent square threads with increase in friction by cosine of half V-angle ($\beta/2$).

3.9.3.1 Lifting Consider a square thread screw used as a jack to lift a load W. Using Eq. (3.33) for $\theta = \pi/2$,

$$\begin{aligned} F &= W\frac{\sin(\phi+\alpha)}{\sin\{\pi/2-(\phi+\alpha)\}} \\ &= W\tan(\alpha+\phi) \end{aligned}$$

If force f is applied at the end of the lever length l, then

$$\begin{aligned} f \times l &= F \times \frac{d}{2} \\ f &= \frac{Wd}{2l}\tan(\phi+\alpha) \end{aligned}$$

Screw efficiency in lifting a load is defined as the ratio of work done in lifting the load per revolution and work done by the applied force per revolution. In one revolution of lifting, the load W travels axial distance equal to lead (l), while the point of applied force F moves the circumferential distance πd, therefore, screw efficiency is given by

$$\begin{aligned} \eta &= \frac{W \times l}{F \times \pi d} \\ &= \frac{\tan\alpha}{\tan(\alpha+\phi)} \end{aligned} \tag{3.35}$$

For maximum screw efficiency,

$$\frac{d\eta}{d\alpha} = 0$$

This results in

$$\alpha = \frac{\pi-\phi}{2}$$

Putting this value in Eq. (3.35), the maximum screw efficiency is found as

$$\eta_{\max} = \frac{1-\sin\phi}{1+\sin\phi}$$

Mechanical advantage is the ratio of weight lifted and applied force:

$$\frac{W}{F} = \frac{2L}{d}\tan(\phi+\alpha)$$

3.9.3.2 Lowering Now, consider a square thread screw used as a jack to lower a load W. Using Eq. (3.34) for $\theta = \pi/2$,

$$\begin{aligned} F &= W\frac{\sin(\phi-\alpha)}{\sin\{\pi/2+(\phi-\alpha)\}} \\ &= W\tan(\phi-\alpha) \end{aligned}$$

If force f is applied at the end of the lever length L, then

$$\begin{aligned} f \times L &= F \times \frac{d}{2} \\ f &= \frac{Wd}{2l}\tan(\alpha-\phi) \end{aligned}$$

This equation indicates that the angle of friction (ϕ) should always be more than the helix angle of the screw (α). Otherwise, the load will slide down of its own weight W. When $\alpha = \phi$, the nut will be on the point of reversing, and using Eq. (3.35) the *screw efficiency in lowering* will be

$$\eta = \frac{\tan\phi}{\tan(2\phi)}$$

For small values of ϕ, $\tan\phi \approx \phi$, hence

$$\eta = \frac{1}{2}$$

Derived with condition $\alpha = \phi$, the above equation indicates that the reversal of nut is avoided if the efficiency of the thread is less than 50% approximately.

3.9.4 Pivots and Collars

Pivots and collars are used to support a rotating shaft subjected to axial loads. *Collars* are provided at any position along the shaft and bears the axial load on a mating surface. *Pivots*, sometimes called footstep bearing, are recesses in which shafts is inserted at one end to bear the axial load [Fig. 3.56]. The surfaces of collars and pivots can be either flat or conical.

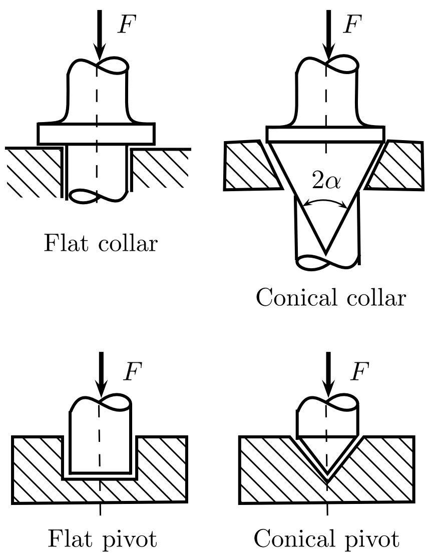


Figure 3.56 | Flat and conical pivots.

The friction torque of a collar or pivot bearing can be determined either by uniform pressure theory or uniform wear theory. Each assumption leads to a different value of friction torque.

3.9.4.1 Uniform Pressure Theory The *uniform pressure theory* assumes that intensity of pressure on the bearing surface is constant. This can be examined in flat and conical collars/pivots:

1. *Flat Collars and Pivots* Consider a flat collar of radii internal radius R_i, external radius R_o, subjected to axial force F. The uniform pressure intensity at any point in the collar is given by

$$p = \frac{F}{\pi\left({R_o}^2 - {R_i}^2\right)}$$

The friction torque will be given by

$$\begin{aligned} T &= \int_{R_i}^{R_o} r \times p \times 2\pi r \times dr \\ &= \int_{R_i}^{R_o} 2p\pi r^2 dr \\ &= \mu F \times \underbrace{\frac{2}{3} \times \frac{{R_o}^3 - {R_i}^3}{{R_o}^2 - {R_i}^2}}_{\bar{R}} \end{aligned} \quad (3.36)$$

Friction torque for flat pivot, based on uniform pressure theory, can be derived using the above expression by taking $R_i = 0$, $R_o = R$:

$$T = \mu F \times \frac{2R}{3}$$

This expression is comparable to Eq. (3.36), where average radius is $\bar{R}$ given by

$$\bar{R} = \frac{2}{3} \times \frac{{R_o}^3 - {R_i}^3}{{R_o}^2 - {R_i}^2}$$

2. *Conical Collars and Pivots* Consider a conical collar of radii internal radius R_i, external radius R_o, half cone angle α, subjected to axial force F. The uniform pressure intensity at any point in the collar is given by

$$F = \int_{R_i}^{R_o} p \times 2\pi r \times \frac{dr}{\sin\alpha}$$

$$p = \frac{F \sin\alpha}{\pi\left({R_o}^2 - {R_i}^2\right)}$$

The friction torque is given by

$$T = \frac{\mu}{\sin\alpha} F \times \underbrace{\frac{2}{3} \times \frac{{R_o}^3 - {R_i}^3}{{R_o}^2 - {R_i}^2}}_{\bar{R}} \quad (3.37)$$

Friction torque for conical pivot, based on uniform pressure theory, can be derived using the above expression by taking $R_i = 0$, $R_o = R$:

$$T = \frac{2}{3} \times \frac{\mu}{\sin\alpha} \times FR$$

This expression is comparable to Eq. (3.37), where the average radius $\bar{R}$ is given by

$$\bar{R} = \frac{2}{3} \times \frac{{R_o}^3 - {R_i}^3}{{R_o}^2 - {R_i}^2}$$

3.9.4.2 Uniform Wear Theory The *uniform wear theory* assumes the uniform wearing of the bearing surface. For this, the intensity of pressure should be inversely proportional to the elemental areas. In this case, for the two locations at r_1 and r_2 with the same width b, the pressure intensities are given by

$$p_1 \times 2\pi r_1 b = p_2 \times 2\pi r_2 b$$
$$p_1 r_1 = p_2 r_2$$

Hence

$$pr = \text{constant}$$

This can be examined in flat and conical collars/pivots:

1. *Flat Collars and Pivots* By this theory, for a flat collar of radii internal radius R_i, external radius R_o, subjected to axial force F, the pressure intensity p at any radius r will be given by

$$p = \frac{F}{2\pi r\,(R_o - R_i)}$$

The friction torque will be

$$\begin{aligned} T &= \int_{R_i}^{R_o} 2p\pi r^2 dr \\ &= \int_{R_i}^{R_o} 2\frac{F}{2\pi r\,(R_o - R_i)}\pi r^2 dr \\ &= \mu F \times \underbrace{\frac{R_o + R_i}{2}}_{\bar{R}} \end{aligned} \tag{3.38}$$

Friction torque for flat pivot, based on uniform wear theory, can be derived using the above expression by taking $R_i = 0$, $R_o = R$:

$$T = \frac{1}{2}\mu F R$$

This expression is comparable to Eq. (3.38), where the average radius $\bar{R}$ is given by

$$\bar{R} = \frac{R_o + R_i}{2}$$

2. *Conical Collars and Pivots* Similarly, for a conical collar of radii internal radius R_i, external radius R_o, half cone angle α, subjected to axial force F, the pressure intensity p at any radius r will be given by

$$p = \frac{F}{2\pi r\,(R_o - R_i)}$$

The friction torque will be given by

$$T = \int_{R_i}^{R_o} r \times \left(\mu p \times 2\pi r \times \frac{dr}{\sin\alpha}\right) \tag{3.39}$$

$$= \frac{\mu F}{\sin\alpha} \times \frac{R_o + R_i}{2} \tag{3.40}$$

Friction torque for conical pivot can be derived using above expression by taking $R_i = 0$, $R_o = R$:

$$T = \frac{\mu}{\sin\alpha} \times F \times \frac{R}{2}$$

This expression is comparable to Eq. (3.40), where average radius is $\bar{R}$ given by

$$\bar{R} = \frac{R_o + R_i}{2}$$

In the above cases, the effect of half cone angle α is to increase the effective coefficient of friction as

$$\mu' = \frac{\mu}{\sin\alpha}$$

Also, by keeping $\alpha = \pi/2$ ($\sin\alpha = 1$) on the expressions for conical bearings (both collar and pivot), the expressions for flat bearings are derived.

The expressions obtained by uniform pressure theory and uniform wear theory give different values. In all the above cases, uniform wear theory gives smaller friction torque than that by uniform pressure theory.

3.9.5 Friction Clutches

A *clutch* is a device used to transmit the rotary motion from one shaft to another. In *friction clutches*, the connection of the two shafts is affected by friction between the two mating concentric surfaces when pressed against each other.

In case of multi-plate clutch having n_1 and n_2 plates on driving and driven shafts, the number of friction surfaces shall be given by

$$n = n_1 + n_2 - 1$$

The expressions of friction torque derived for pivots and collars can be used to determine the maximum torque and power transmission. In design of clutches, the objective is to make the clutch capable for maximum torque transmission, therefore, using *uniform wear theory* gives safer values.

3.10 GOVERNOR

Governor is a device used to maintain the speed of an engine within specified limits when the engine works in varying of load. This can be distinguished from that of a flywheel which controls energy fluctuations per cycle (Table 3.2). The functioning of flywheel is independent of speed. The operation of a governor is intermittent while that of a flywheel is continuous.

3.10.1 Types of Governors

Based on the source of controlling force, the governors are of two types:

Table 3.2 | Governor versus flywheel

	Governor	Flywheel
1.	Provided on prime-movers	Provided in engine and machines
2.	Regulates supply of fuel	Stores mechanical energy
3.	Takes care of long range variation in load	Take care of variation in the cycle
4.	Works only when load changes	Works in each cycle

1. *Centrifugal Governors* In *centrifugal governors*, two or more masses, known as *governor balls*, are caused to revolve about the axis of a shaft, which is driven through suitable gearing from the engine crankshaft. Each ball is acted upon by a force which acts in the radially inward direction, and is provided by a deadweight, a spring or a combination of two. This force is termed as the *controlling force* and it must increase in magnitude as the distance of the ball from the axis of rotation increases. The inward or outward movement of the balls is transmitted by the governor mechanism to the valve which controls the amount of energy supplied to the engine.

2. *Inertia Governors* In inertia governors, the balls are so arranged that the inertia forces caused by an *angular acceleration* or retardation of the governor shaft tend to alter their position. The obvious advantage of this type of governor lies in its more rapid response to the effect of a change of load. This advantage is offset, however, by the practical difficulty of arranging for the complete balance of the revolving parts of the governor. For this reason, centrifugal governors are much more frequently used than are inertia governors, and thus, only the former type will be dealt with here.

The concept of some important governors is explained as follows

3.10.1.1 Watt Governor *Watt governor*[8], although now obsolete, is interesting as being the forerunner of the later examples of governors. It consists of two balls attached to the spindle through four arms. The two upper arms meet at the pivot on the spindle axis. The lower arms are connected at a sleeve by pin joints. The movement of the sleeve is restricted by stops [Fig. 3.57].

[8]Watt governor is the original form of governor as used by Watt on some of his early steam engines.

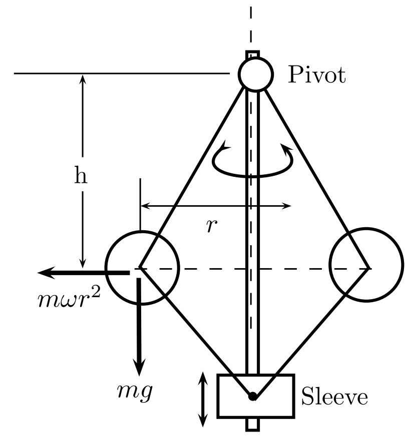


Figure 3.57 | Watt governor.

Consider the situation when the rotation speed is N rpm ($=\omega$ rad/s) and the balls are located at radius r and height h measured from pivot. The centrifugal forces acting on all four balls is $m\omega^2 r$. The weight of the balls is mg. Taking moment of these forces acting on the balls about the pivot gives

$$m\omega^2 r \times h - mg \times r = 0$$
$$h = \frac{g}{\omega^2}$$

Using this equation, the height h can be expressed in terms of speed N (rpm) as

$$h = \frac{895}{N^2}$$

Differentiating the above equation w.r.t. N, the change in height is found to be related with change in speed as

$$\delta h \propto -\frac{\delta N}{N^3}$$

Therefore, with increasing speeds, δh becomes insignificant, and governor stops functioning. Therefore, Watt governors are suitable for slow speed engines only.

3.10.1.2 Porter Governor The porter governor is a modified Watt governor in which a heavy central mass M is placed to the sleeve. The action is exactly the same as that of the Watt governor [Fig. 3.58].

With equilibrium equation of balls, the expression for height h can be derived as

$$h = \frac{mg + (Mg \pm F)(1+q)/2}{mg} \times \frac{895}{N^2}$$

where F is the frictional force on movement of central mass, and q is a constant defined as

$$q = \frac{\tan\alpha}{\tan\beta}$$

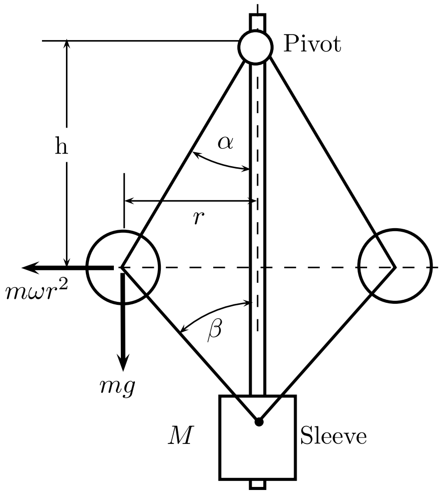


Figure 3.58 | Porter governor.

The quantity q will have a different values for each radius of rotation of the governor balls, unless the upper and lower arms are of equal length.

3.10.1.3 Proel Governor A Proel governor is similar to the Porter governor in that it is has a heavily weighted sleeve, but differs from it in the arrangement of balls [Fig. 3.59]. These are carried on the extension of the lower arm instead of being carried out at the junction of the upper and lower arms. The action of this governor is similar to Watt governor.

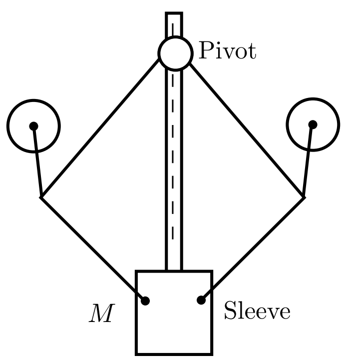


Figure 3.59 | Proel governor.

3.10.1.4 Hartnell Governor The Hartnell governor is of the spring loaded type Fig. 3.60. It consists of two bell crank levers pivoted at points offset to central axis. The frame is attached to the governor spindle and rotates with it. Each ball lever carries a ball at the end of vertical arms, and roller at the other end of horizontal arm. A helical spring provides equal downward forces on the two rollers through sleeve.

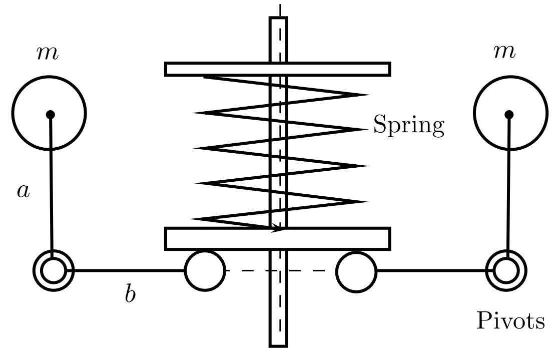


Figure 3.60 | Hartnell governor.

Let r_1 and r_2 be the radius of balls at two speeds ω_1 and ω_2. The centrifugal forces acting on the balls in both conditions will be

$$F_1 = m{\omega_1}^2 r_1$$
$$F_2 = m{\omega_2}^2 r_2$$

Taking moments of forces acting on the balls w.r.t. pivots for both situations separately,

$$F_1 a_1 = \frac{Mg + f_s + f}{2} b_1 - mgc_1$$
$$F_2 a_2 = \frac{Mg + f_s + f}{2} b_2 - mgc_2$$

where f_s is spring force, f is friction force, and c_1 and c_2 are the offsets of the balls. Taking,

$$a_1 \approx a_2 \approx a, \quad b_1 \approx b_2 \approx b, \quad c_1 = c_2 = 0$$

Therefore,

$$F_1 a = \frac{Mg + f_{s_1} + f \times b}{2}$$
$$F_2 a = \frac{Mg + f_{s_2} + f \times b}{2}$$

Subtracting the above two equations,

$$(F_2 - F_1)\, a = (f_{s_2} - f_{s_1})\, \frac{b}{2}$$

The difference in spring force is

$$f_{s_2} - f_{s_1} = \frac{2a}{b}(F_2 - F_1)$$

3.10.2 Sensitiveness and Stability

The following are the important terms related to sensitiveness and stability of governors:

1. <u>*Sensitiveness*</u> Sensitiveness of a governor is correctly defined as the ratio of the difference

between the maximum and minimum equilibrium speeds to the mean equilibrium speed:

$$\text{Sensitiveness} = \frac{N_1 - N_2}{N}$$

where N_1 and N_2 shows the speed range between which the governor is insensitive, and N is the mean equilibrium speed. This is also referred as the *coefficient of insensitiveness* as a measure of insensitiveness of a governor.

2. Isochronism A governor is said to be *isochronous* when it has the same equilibrium speed for all the positions of sleeve or the balls; any change of speed results in moving the balls into extreme positions. Thus, an isochrounous governor will have infinite sensitiveness.

3. Stability A governor is said to be stable when for each speed within the working range, there is only one radius of rotation of the governor balls at which the governor is in equilibrium.

4. Hunting *Sensitiveness* of a governor is a desirable quality. However, if a governor is too sensitive, it can fluctuate continuously because when the load on the engine falls, the sleeve rises rapidly to a maximum position. This shuts off the fuel supply. If the frequency of fluctuations in engine speed happens to coincide with the natural frequency of oscillations of the governor, then due to resonance, the amplitude of oscillations becomes very high with the result that the governor tends to intensify the speed variations instead of controlling it. Such a situation is known as *hunting*.

3.10.3 Controlling Force

In a centrifugal governor, the resultant of all external forces which control the movement of the ball can be regarded as a single inward radial force acting at the center of the ball. The variation of this force F with radius of rotation r of the ball, which is necessary to keep the ball in equilibrium at various configurations (i.e. for different values of r). The force F is known as *controlling force* which is function of single variable r:

$$F = f(r)$$

Let the ball rotates at a speed ω then centrifugal force needed for maintaining the radius r is $m\omega^2 r$. So, for given value of ω

$$F \propto r$$

The governor is said to be stable when slope of speed curve is less than that of controlling force curve. For a given governor to be stable at all radii (i.e. throttling can be controlled and will bring speed to the desired value

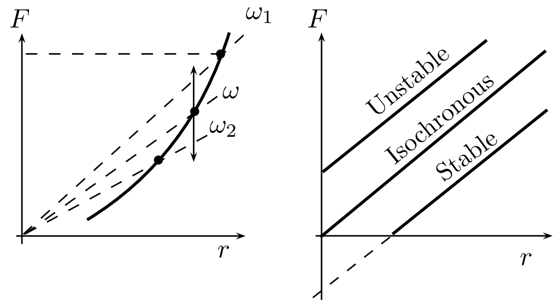


Figure 3.61 | Controlling force diagram.

without hunting), the controlling force curve should pass $r = +\text{ve}$ or 0 [Fig. 3.61]. For stability,

$$\frac{F}{r} < \frac{\partial F}{\partial r}$$

The friction force at the sleeve gives rise to the insensitiveness in the governor. At any given radius r, there will be two different speeds one being when sleeve moves up and other being when sleeve moves down. Fig. 3.61 shows that when the speed increases from ω (N) to ω_1 (N_1), r increases to r_1, and F increases to F_1 and sleeve closes the throttling value. Similarly, when $\omega_1 \to \omega_2$ (N_2), and r decreases to r_2.

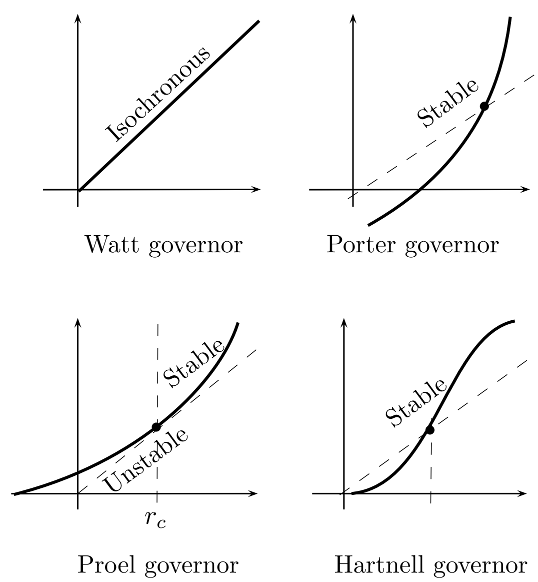


Figure 3.62 | Stability and isochronism of gonvernors

Figure 3.62 depicts the plots of controlling force versus radius of rotation of balls for different types of governors. It is evident that Watt governor is an isochronous governor. Porter governor is a stable governor while Proel governor is stable after a certain radius. Hartnell governor is also a stable governor.