ENTRI

CHAPTER 1

LINEAR ALGEBRA

MATRIX

A set of mn number (real or imaginary) arranged in the form of a rectangular array of m rows and n columns is called an $m \times n$ matrix. An $m \times n$ matrix is usually written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

In compact form, the above matrix is represented by $A = [a_{ij}]_{m\times n}$ or $A = [a_{ij}]$.

The numbers a_{11}, a_{12}, ..., a_{mn} are known as the elements of the matrix A. The element a_{ij} belongs to i^{th} row and j^{th} column and is called the $(ij)^{\text{th}}$ element of the matrix $A = \left[a_{ij}\right]$.

Types of Matrices

1. **Row matrix:** A matrix having only one row is called a row matrix or a row vector. Therefore, for a row matrix, $m = 1$.

   For example, $A = [1\ 2\ -1]$ is a row matrix with $m = 1$ and $n = 3$.

2. **Column matrix:** A matrix having only one column is called a column matrix or a column vector. Therefore, for a column matrix, $n = 1$.

   For example, $A = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ is a column matrix with $m = 3$ and $n = 1$.

3. **Square matrix:** A matrix in which the number of rows is equal to the number of columns, say n, is called a square matrix of order n.

   For example, $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is a square matrix of order 2.

4. **Diagonal matrix:** A square matrix is called a diagonal matrix if all the elements except those in the leading diagonal are zero, i.e. $a_{ij} = 0$ for all $i \neq j$.

For example, $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 10 \end{bmatrix}$ is a diagonal matrix and is denoted by A = diag[1, 5, 10].

5. **Scalar matrix:** A matrix $A = [a_{ij}]_{n\times n}$ is called a scalar matrix if
(a) $a_{ij} = 0$, for all $i \neq j$.
(b) $a_{ii} = c$, for all i, where $c \neq 0$.

For example, $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ is a scalar matrix of order 5.

6. **Identity or unit matrix:** A square matrix $A = [a_{ij}]_{n\times n}$ is called an identity or unit matrix if
(a) $a_{ij} = 0$, for all $i \neq j$.
(b) $a_{ij} = 1$, for all i.

For example, $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is an identity matrix of order 2.

7. **Null matrix:** A matrix whose all the elements are zero is called a null matrix or a zero matrix.

For example, $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a null matrix of order 2×2.

8. **Upper triangular matrix:** A square matrix $A = [a_{ij}]$ is called an upper triangular matrix if $a_{ij} = 0$ for $i > j$.

For example, $A = \begin{bmatrix} 1 & 2 & 6 & 3 \\ 0 & 5 & 7 & 4 \\ 0 & 0 & 9 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ is an upper triangular matrix.

9. **Lower triangular matrix:** A square matrix $A = [a_{ij}]$ is called a lower triangular matrix if $a_{ij} = 0$ for $i < j$.

For example, $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 5 & 0 & 0 \\ 6 & 2 & 9 & 0 \\ 3 & 7 & 4 & 3 \end{bmatrix}$ is a lower triangular matrix.

Types of a Square Matrix

1. **Nilpotent matrix:** A square matrix A is called a nilpotent matrix if there exists a positive integer n such that $A^n = 0$. If n is least positive integer such that $A^n = 0$, then n is called index of the nilpotent matrix A.

2. **Symmetrical matrix:** It is a square matrix in which $a_{ij} = a_{ji}$ for all i and j. A symmetrical matrix is necessarily a square one. If A is symmetric, then $A^T = A$.

For example, $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 4 & 6 \end{bmatrix}$.

3. **Skew-symmetrical matrix:** It is a square matrix in which $a_{ij} = -a_{ji}$ for all i and j. In a skew-symmetrical matrix, all elements along the diagonal are zero.

For example, $\begin{bmatrix} 0 & 2 & 3 \\ 2 & 0 & 4 \\ 3 & 4 & 0 \end{bmatrix}$.

4. **Hermitian matrix:** It is a square matrix A in which $(i, j)^{\text{th}}$ element is equal to complex conjugate of the $(j, i)^{\text{th}}$ element, i.e. $a_{ij} = \bar{a}_{ji}$ for all i and j. A necessary condition for a matrix A to be Hermitian is that $A = A^\theta$, where A^θ is transposed conjugate of A.

For example, $\begin{bmatrix} 1 & 1+4i & 2+3i \\ 1-4i & 2 & 5+i \\ 2-3i & 5-i & 4 \end{bmatrix}$.

5. **Skew-Hermitian matrix:** It is a square matrix $A = [a_{ij}]$ in which $a_{ij} = -\bar{a}_{ij}$ for all i and j.

The diagonal elements of a skew-Hermitian matrix must be pure imaginary numbers or zeroes. A necessary and sufficient condition for a matrix A to be skew-Hermitian is that

$$A^\theta = -A$$

6. **Orthogonal matrix:** A square matrix A is called orthogonal matrix if $AA^T = A^TA = I$.

For example, if

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \text{ and } A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

then $AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$.

Equality of a Matrix

Two matrices $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{x\times y}$ are equal if

1. $m = x$, i.e. the number of rows in A equals the number of rows in B.
2. $n = y$, i.e. the number of columns in A equals the number of columns in B.
3. $a_{ij} = b_{ij}$ for $i = 1, 2, 3, \ldots, m$ and $j = 1, 2, 3, \ldots, n$.

Addition of Two Matrices

Let A and B be two matrices, each of order $m \times n$. Then their sum $(A + B)$ is a matrix of order $m \times n$ and is obtained by adding the corresponding elements of A and B.

Thus, if $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ are two matrices of the same order, their sum $(A + B)$ is defined to be the matrix of order $m \times n$ such that

$$(A + B)_{ij} = a_{ij} + b_{ij} \text{ for } i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n$$

The sum of two matrices is defined only when they are of the same order.

For example, if $A = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 8 \\ 1 & 2 \end{bmatrix}$

Hence, $A + B = \begin{bmatrix} 9 & 9 \\ 4 & 7 \end{bmatrix}$

However, addition of $\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \end{bmatrix}$ is not possible.

Some of the important properties of matrix addition are:

1. **Commutativity:** If A and B are two $m \times n$ matrices, then $A + B = B + A$, i.e. matrix addition is commutative.
2. **Associativity:** If A, B and C are three matrices of the same order, then
$$(A + B) + C = A + (B + C)$$
i.e.matrix addition is associative.
3. **Existence of identity:** The null matrix is the identity element for matrix addition. Thus, $A + O = A = O + A$
4. **Existence of inverse:** For every matrix $A = [a_{ij}]_{m\times n}$, there exists a matrix $[a_{ij}]_{m\times n}$, denoted by $-A$, such that $A + (-A) = O = (-A) + A$
5. **Cancellation laws:** If A, B and C are matrices of the same order, then
$$A + B = A + C \Rightarrow B = C$$
$$\mathrm{B} + A = C + A \Rightarrow B = C$$

Multiplication of Two Matrices

If we have two matrices A and B, such that

$$A = [a_{ij}]_{m\times n} \quad \text{and} \quad B = [b_{ij}]_{x\times y}, \text{ then}$$

$A \times B$ is possible only if $n = x$, i.e. the columns of the pre-multiplier is equal to the rows of the post multiplier. Also, the order of the matrix formed after multiplying will be $m \times y$.

$$(AB)_{ij} = \sum_{r=1}^{n} a_{ir}b_{rj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

For example, if $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$

Then $A \times B = \begin{bmatrix} 1\times4+2\times2 & 1\times3+2\times1 \\ 3\times4+4\times2 & 3\times3+4\times1 \end{bmatrix}$

$C = A \times B = \begin{bmatrix} 8 & 5 \\ 20 & 13 \end{bmatrix}$

Some important properties of matrix multiplication are:

1. Matrix multiplication is not commutative.
2. Matrix multiplication is associative, i.e. $(AB)C = A(BC)$.
3. Matrix multiplication is distributive over matrix addition, i.e. $A(B + C) = AB + AC$.
4. If A is an $m\times n$ matrix, then $I_mA = A = AI_n$.
5. The product of two matrices can be the null matrix while neither of them is the null matrix.

Multiplication of a Matrix by a Scalar

If $A = [a_{ij}]$ be an $m \times n$ matrix and k be any scalar constant, then the matrix obtained by multiplying every element of A by k is called the scalar multiple of A by k and is denoted by kA.

For example, if $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 7 & 3 \\ 4 & 9 & 8 \end{bmatrix}$ and $k = 2$.

$$\Rightarrow kA = \begin{bmatrix} 2 & 6 & 2 \\ 2 & 14 & 6 \\ 3 & 18 & 16 \end{bmatrix}$$

Some of the important properties of scalar multiplication are:

1. $k(A + B) = kA + kB$
2. $(k + l) \cdot A = kA + lA$
3. $(kl) \cdot A = k(lA) = l(kA)$
4. $(-k) \cdot A = -(kA) = k(-A)$
5. $1 \cdot A = A$
6. $-1 \cdot A = -A$

Here A and B are two matrices of same order and k and l are constants.

If A is a matrix and $A^2 = A$, then A is called idempotent matrix. If A is a matrix and satisfies $A^2 = I$, then A is called involuntary matrix.

Transpose of a Matrix

Consider a matrix A, then the matrix obtained by interchanging the rows and columns of A is called its transpose and is represented by A^T.

For example, if $A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 9 & 8 \\ 7 & 6 & 4 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 3 & 7 \\ 2 & 9 & 6 \\ 5 & 8 & 4 \end{bmatrix}$.

Some of the important properties of transpose of a matrix are:

1. For any matrix A, $(A^T)^T = A$
2. For any two matrices A and B of the same order
$$(A + B)^T = A^T + B^T$$
3. If A is a matrix and k is a scalar, then
$$(kA)^T = k(A^T)$$
4. If A and B are two matrices such that AB is defined, then
$$(AB)^T = B^T A^T$$

Adjoint of a Square Matrix

Let $A = [a_{ij}]$ be a square matrix of order n and let C_{ij} be the cofactor of a_{ij} in A. Then the transpose of the matrix of cofactors of elements of A is called the adjoint of A and is denoted by adj A.

Thus, adj $A = [C_{ij}]^T \Rightarrow (\text{adj } A)_{ij} = C_{ji}$ = cofactor of a_{ji} in A.

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

then $\text{adj}(A) = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix}$

Inverse of a Matrix

A square matrix of order n is invertible if there exists a square matrix B of the same order such that
$$AB = I_n = BA$$

In the above case, B is called the inverse of A and is denoted by A^{-1}.
$$A^{-1} = \frac{(\text{adj } A)}{|A|}$$

Some of the important properties of inverse of a matrix are:

1. A^{-1} exists only when A is non-singular, i.e. $|A| \neq 0$.
2. The inverse of a matrix is unique.
3. Reversal laws: If A and B are invertible matrices of the same order, then
$$(AB)^{-1} = B^{-1}A^{-1}$$
4. If A is an invertible square matrix, then $(A^T)^{-1} = (A^{-1})^T$
5. The inverse of an invertible symmetric matrix is a symmetric matrix.
6. Let A be a non-singular square matrix of order n. Then
$$|\text{adj } A| = |A|^{n-1}$$
7. If A and B are non-singular square matrices of the same order, then
$$\text{adj } (AB) = (\text{adj } B)(\text{adj } A)$$
8. If A is an invertible square matrix, then
$$\text{adj } A^T = (\text{adj } A)^T$$
9. If A is a non-singular square matrix, then
$$\text{adj } (\text{adj } A) = |A|^{n-2} A$$
10. If A is a non-singular matrix, then
$$|A^{-1}| = |A|^{-1}, \text{ i.e. } |A^{-1}| = \frac{1}{|A|}$$
11. Let A, B and C be three square matrices of same type and A be a non-singular matrix. Then
$$AB = AC \Rightarrow B = C$$
and
$$BA = CA \Rightarrow B = C$$

Rank of a Matrix

The column rank of matrix A is the maximum number of linearly independent column vectors of A. The row rank of A is the maximum number of linearly independent row vectors of A.

In linear algebra, column rank and row rank are always equal. This number is simply called rank of a matrix.

The rank of a matrix A is commonly denoted by rank (A). Some of the important properties of rank of a matrix are:

1. The rank of a matrix is unique.
2. The rank of a null matrix is zero.
3. Every matrix has a rank.
4. If A is a matrix of order $m \times n$, then rank $(A) \leq m \times n$ (smaller of the two)
5. If rank $(A) = n$, then every minor of order $n + 1$, $n + 2$, etc., is zero.
6. If A is a matrix of order $n \times n$, then A is non-singular and rank $(A) = n$.
7. Rank of $I_A = n$.
8. A is a matrix of order $m \times n$. If every k^{th} order minor $(k < m, k < n)$ is zero, then
$$\text{rank } (A) < k$$
9. A is a matrix of order $m \times n$. If there is a minor of order $(k < m, k < n)$ which is not zero, then
$$\text{rank } (A) \geq k$$
10. If A is a non-zero column matrix and B is a non-zero row matrix, then rank $(AB) = 1$.
11. The rank of a matrix is greater than or equal to the rank of every sub-matrix.
12. If A is any n-rowed square matrix of rank, $n - 1$, then
$$\text{adj } A \neq 0$$
13. The rank of transpose of a matrix is equal to rank of the original matrix.
$$\text{rank } (A) = \text{rank } (A^T)$$

14. The rank of a matrix does not change by pre-multiplication or post-multiplication with a non-singular matrix.
15. If $A - B$, then rank (A) = rank (B).
16. The rank of a product of two matrices cannot exceed rank of either matrix.

$$\text{rank } (A \times B) \leq \text{rank } A$$

or

$$\text{rank } (A \times B) \leq \text{rank } B$$

17. The rank of sum of two matrices cannot exceed sum of their ranks.
18. Elementary transformations do not change the rank of a matrix.

DETERMINANTS

Every square matrix can be associated to an expression or a number which is known as its determinant. If $A = [a_{ij}]$ is a square matrix of order n, then the determinant of A is denoted by det A or $|A|$. If $A = [a_{11}]$ is a square matrix of order 1, then determinant of A is defined as

$$|A| = a_{11}$$

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{23} \end{bmatrix}$ is a square matrix of order 2, then determinant of A is defined as

$$|A| = a_{11}a_{23} - a_{12}a_{21}$$

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a square matrix of order 3, then determinant of A is defined as

$$A| = a_{11}\ (a_{22}a_{33} - a_{23}a_{32}) - a_{21}\ (a_{12}a_{33} - a_{13}a_{32}) + a_{21}\ (a_{12}a_{23} - a_{13}a_{22})$$

or

$$|A| = a_{11}\ (a_{22}a_{33} - a_{23}a_{32}) - a_{12}\ (a_{21}a_{33} - a_{23}a_{31}) + a_{13}\ (a_{21}a_{32} - a_{22}a_{31})$$

For example, determinants of the matrices $[1], \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$ and $\begin{bmatrix} 1 & 7 & 1 \\ 2 & -7 & 3 \\ 5 & 9 & 8 \end{bmatrix}$ will be, respectively,

$$\Delta = 1$$
$$\Delta = (2 \times 5) - (1 \times 3) = 7$$
$$\Delta = 1[(-7 \times 8) - (3 \times 9)] - 2\ [(7 \times 8) - (1 \times 9)] + 5\ [(7 \times 3) - (-7 \times 1)]$$
$$= -83 - 94 + 140 = -177 + 40$$
$$= -37$$

Minors

The minor M_{ij} of $A = [a_{ij}]$ is the determinant of the square sub-matrix of order $(n - 1)$ obtained by removing i^{th} row and j^{th} column of the matrix A.

For example, say $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$, then minors of A will be

$$M_{11} = 4$$
$$M_{12} = 2$$
$$M_{21} = 3$$
$$M_{22} = 1$$

Say $$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 1 \\ -4 & 4 & 7 \end{bmatrix}$$

$$M_{11} = \begin{bmatrix} 5 & 1 \\ 4 & 7 \end{bmatrix} = 35 - 4 = 31$$

$$M_{12} = \begin{bmatrix} 3 & 1 \\ -4 & 7 \end{bmatrix} = 21 - (-4) = 25$$

$$M_{13} = \begin{bmatrix} 3 & 5 \\ -4 & 4 \end{bmatrix} = 12 - (-20) = 32$$

$$M_{21} = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix} = 14 - 12 = 2$$

$$M_{22} = \begin{bmatrix} 1 & 3 \\ -4 & 7 \end{bmatrix} = 7 - (-12) = 19$$

$$M_{23} = \begin{bmatrix} 1 & 2 \\ -4 & 4 \end{bmatrix} = 4 - (-8) = 12$$

$$M_{31} = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} = 2 - 15 = -13$$

$$M_{32} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} = 1 - 9 = -8$$

$$M_{33} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = 5 - 6 = -1$$

Cofactors

The cofactor C_{ij} of $A = [a_{ij}]$ is equal to $(-1)^{i+j}$ times the determinant of the sub-matrix of order $(n - 1)$ obtained by leaving i^{th} row and j^{th} column of A.

For example, say $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 1 \\ -4 & 4 & 7 \end{bmatrix}$.

Cofactors of A will be

$$C_{11} = (-1)^{1+1} \begin{bmatrix} 5 & 1 \\ 4 & 7 \end{bmatrix} = 31$$

$$C_{12} = (-1)^{1+2} \begin{bmatrix} 3 & 1 \\ -4 & 7 \end{bmatrix} = -25$$

$$C_{13} = (-1)^{1+3} \begin{bmatrix} 3 & 5 \\ -4 & 4 \end{bmatrix} = 32$$

$$C_{21} = (-1)^{2+1} \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix} = -2$$

$$C_{22} = (-1)^{2+2} \begin{bmatrix} 1 & 3 \\ -4 & 7 \end{bmatrix} = 19$$

$$C_{23} = (-1)^{2+3} \begin{bmatrix} 1 & 2 \\ -4 & 4 \end{bmatrix} = -12$$

$$C_{31} = (-1)^{3+1}\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} = -13$$

$$C_{32} = (-1)^{3+2}\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} = 8$$

$$C_{33} = (-1)^{3+3}\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = -1$$

Some of the important properties of determinants are:

1. Sum of the product of elements of any row or column of a square matrix $A = [a_{ij}]$ of order n with their cofactors is always equal to its determinant.

$$\sum_{i=1}^{n} a_{ij}c_{ij} = |A| = \sum_{j=1}^{n} a_{ij}c_{ij}$$

2. Sum of the product of elements of any row or column of a square matrix $A = [a_{ij}]$ of order n with the cofactors of the corresponding elements of other row or column is zero.

$$\sum_{i=1}^{n} a_{ij}c_{ik} = 0 = \sum_{j=1}^{n} a_{ij}c_{kj}$$

3. For a square matrix $A = [a_{ij}]$ of order n, $|A| = |A^T|$.
4. Consider a square matrix $A = [a_{ij}]$ of order $n \geq 2$ and B obtained from A by interchanging any two rows or columns of A, then $|B| = -A$.
5. For a square matrix $A = [a_{ij}]$ of order $(n \geq 2)$, if any two rows or columns are identical, then its determinant is zero, i.e. $|A| = 0$.
6. If all the elements of a square matrix $A = [a_{ij}]$ of order n are multiplied by a scalar k, then the determinant of new matrix is equal to $k|A|$.
7. Let A be a square matrix such that each element of a row or column of A is expressed as the sum of two or more terms. Then $|A|$ can be expressed as the sum of the determinants of two or more matrices of the same order.
8. Let A be a square matrix and B be a matrix obtained from A by adding to a row or column of A a scalar multiple of another row or column of A, then $|B| = |A|$.
9. Let A be a square matrix of order n (≥ 2) and also a null matrix, then $|A| = 0$.
10. Consider $A = [a_{ij}]$ as a diagonal matrix of order n (≥ 2), then
$$|A| = a_{11} \times a_{22} \times a_{33} \times \cdots \times a_{nn}$$
11. Suppose A and B are square matrices of same order, then
$$|AB| = |A| \cdot |B|$$

SOLUTIONS OF SIMULTANEOUS LINEAR EQUATIONS

Suppose we have a system of 'i' linear equation in 'j' unknowns, such as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{ij}x_j &= b_1 \\ a_{221}x_1 + a_{22}x_2 + \cdots + a_{2j}x_j &= b_2 \\ \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad & \vdots \\ \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad & \vdots \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{ij}x_j &= b_i \end{aligned}$$

The above system of equations can be written in the matrix form as follows:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{ij} \\ a_{21} & a_{22} & \cdots & a_{2j} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_j \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_j \end{bmatrix}$$

The equation can be represented by the form $AX = B$.

where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{ij} \\ a_{21} & a_{22} & \cdots & a_{2j} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} \end{bmatrix}$ is of the order of $i \times j$,

$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_j \end{bmatrix}$ is of the order of $j \times 1$ and

$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_i \end{bmatrix}$ is of the order of $i \times 1$.

$[A]_{i \times j}$ is called the coefficient matrix of system of linear equations.

For example,
$$\begin{aligned} 2x + 3y - z &= 2 \\ 3x + y + 2z &= 1 \\ 4x - 7y + 5z &= -7 \end{aligned}$$

The above equation can be expressed in the matrix form as

$$\begin{bmatrix} 2 & 3 & -1 \\ 3 & 1 & 2 \\ 4 & -7 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -7 \end{bmatrix}$$

A consistent system is a system of equations that has one or more solutions.

An inconsistent system is a system of equations that has no solutions.

For example, consider

$$\begin{aligned} x + y &= 4 \\ 2x + 3y &= 9 \end{aligned}$$

The above system of equation is consistent as it has the solution, $x = 3$ and $y = 1$.

Now, consider

$$x + y = 4$$
$$2x + 2y = 6$$

The above system of equation is inconsistent as it has no solution for both equations simultaneously.

A homogeneous system is a system when $B = 0$ in the equation $AX = B$.

A non-homogeneous system is a system when $B \neq 0$ in the equation $AX = B$.

For example, consider

$$3x + y = 0$$
$$6x + 2y = 0$$

Such a system is a homogeneous system.

Now, consider

$$3x + y = 1$$
$$6x + 2y = 2$$

Such a system is a non-homogeneous system.

Solution of Homogeneous System of Linear Equations

As already discussed, for a homogeneous system of linear equation with 'j' unknowns,

$$AX = B \text{ becomes}$$
$$AX = 0 \ (\because\ B = 0)$$

There are two cases that arise for homogeneous systems:

1. Matrix A is non-singular or $|A| \neq 0$.

   The solution of the homogeneous system in the above equation has a unique solution, $X = 0$, i.e. $x_1 = x_2 = \cdots = x_j = 0$.

2. Matrix A is singular or $|A| = 0$, then it has infinite many solutions. To find the solution when $|A| = 0$, put $z = k$ (where k is any real number) and solve any two equations for x and y using the matrix method. The values obtained for x and y with $z = k$ is the solution of the system.

   For example, given the following system of homogeneous equation

$$2x + y + 3z = 0$$
$$3x - y + z = 0$$
$$-x - 2y + 3z = 0$$

   This system can be rewritten as

$$\begin{bmatrix} 2 & 1 & 3 \\ 3 & -1 & 1 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $AX = 0$

$$A = \begin{vmatrix} 2 & 1 & 3 \\ 3 & -1 & 1 \\ -1 & -2 & 3 \end{vmatrix} = (2(-3+2) - 3(3+6) - (1+3))$$
$$= -2 - 27 - 4 = -33 \neq 0$$

As $|A| \neq 0$, given system of equations has the solution $x = y = z = 0$.

Solution of Non-Homogeneous System of Simultaneous Linear Equations

Non-homogeneous equations have already been discussed in the previous sections. Also, the methods to solve homogeneous system of equations were also discussed in the previous section.

In this section, we will discuss the method to solve a non-homogeneous system of simultaneous linear equations. Please note the number of unknowns and the number of equations.

1. Given that A is a non-singular matrix, then a system of equations represented by $AX = B$ has the unique solution which can be calculated by $X = A^{-1} B$.
2. If $AX = B$ is a system with linear equations equal to the number of unknowns, then three cases arise:
   (a) If $|A| \neq 0$, system is consistent and has a unique solution given by $X = A^{-1} B$.
   (b) If $|A| = 0$ and $(\text{adj } A)B = 0$, system is consistent and has infinite solutions.
   (c) If $|A| = 0$ and $(\text{adj } A)B \neq 0$, system is inconsistent.

For example, consider the following system of equations:

$$x + 2y + z = 7$$
$$x + 3z = 11$$
$$2x - 3y = 1$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \\ 1 \end{bmatrix}$$

$$\Rightarrow AX = B, \text{ where } A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{vmatrix} = 1(0+9) - 1(0+3) + 2(6)$$
$$= 9 - 3 + 12 = 18 \neq 0$$

So, the given system of equations has a unique solution given by $X = A^{-1} B$. Now, calculate the adjoint of matrix.

$$\text{adj}\,A = \begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj}\,A = \frac{1}{18}\begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}$$

$$X = A^{-1}B$$

$$X = \frac{1}{18}\begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}\begin{bmatrix} 7 \\ 11 \\ 1 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 63-33+6 \\ 42-22-2 \\ -21+77-2 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 36 \\ 18 \\ 54 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$\Rightarrow x = 2,\ y = 1,\ z = 3$ is the unique solution of the given set of equations.

Cramer's Rule

Suppose we have the following system of linear equations:

$$a_1x + b_1y + c_1z = k_1$$
$$a_2x + b_2y + c_2z = k_2$$
$$a_3x + b_3y + c_3z = k_3$$

Now, if

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$$

$$\Delta_1 = \begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix} \neq 0$$

$$\Delta_2 = \begin{vmatrix} a_1 & k_1 & c_1 \\ a_2 & k_2 & c_2 \\ a_3 & k_3 & c_3 \end{vmatrix} \neq 0$$

$$\Delta_3 = \begin{vmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{vmatrix} \neq 0$$

Thus, the solution of the system of equations is given by

$$x = \frac{\Delta_1}{\Delta}$$

$$y = \frac{\Delta_2}{\Delta}$$

$$z = \frac{\Delta_3}{\Delta}$$

AUGMENTED MATRIX

Consider the following system of equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

This system can be represented as $AX = B$.

where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

The matrix $[A|B] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$ is called augmented matrix.

GAUSS ELIMINATION METHOD

Consider the following system of equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ \vdots \qquad \vdots \qquad\quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= 0 \end{aligned}$$

In the matrix form, the above equations can be represented as $AX = B$. We consider two particular cases:

Case 1: Suppose the coefficient matrix A is such that all elements above the leading diagonal are zero. Then A is a lower triangular matrix and the system can be written as follows:

$$\begin{aligned} a_{11}x_1 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

The system given above can be solved using forward substitution.

$$\therefore x_1 = \frac{b_1}{a_{11}}; x_2 = \frac{b_2 - a_{21}x_1}{a_{22}}; x_3 = \frac{b_3 - (a_{31}x_1 + a_{32}x_2)}{a_{33}}$$

and so on.

Case 2: Suppose the coefficient matrix A is such that all elements below the leading diagonal are zero. Then A is an upper triangular matrix and the system can be written in the following form:

$$\begin{aligned} &a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ &a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ &\vdots \\ &a_{nn}x_n = b_n \end{aligned}$$

The system given above can be solved using backward substitution.

$$\therefore x_3 = \frac{b_3}{a_{33}}; x_2 = \frac{b_2 - a_{23}x_3}{a_{22}}; x_1 = \frac{b_1 - (a_{12}x_2 + a_{13}x_3)}{a_{11}}$$

and so on.

Gauss elimination method is a standard elimination method for solving linear systems that proceeds systematically irrespective of particular features of the coefficients.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

The first equation of the above system is called the pivotal equation and the leading coefficient a_{11} is called first pivot.

Step 1: Form the augmented matrix of the above system.

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

Step 2: Performing the following operations $R_2 \to R_2 - (a_{21}/a_{11})R_1$ and $R_3 \to R_3 - (a_{31}/a_{11})R_1$, we get

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a'_{22} & a'_{23} & b'_2 \\ 0 & a'_{32} & a'_{33} & b'_3 \end{array}\right]$$

where

$$\begin{aligned} a'_{22} &= a_{22} - (a_{21}/a_{11})a_{12} \\ a'_{23} &= a_{23} - (a_{21}/a_{11})a_{13} \\ a'_{32} &= a_{32} - (a_{31}/a_{11})a_{12} \\ a'_{33} &= a_{33} - (a_{31}/a_{11})a_{13} \\ b'_2 &= b_2 - (a_{21}/a_{11})b_1 \\ b'_3 &= b_3 - (a_{31}/a_{11})b_1 \end{aligned}$$

Here $-a_{21}/a_{11}$ and $-a_{31}/a_{11}$ are called the multipliers for the first stage of elimination. It is clear that we have assumed $a_{11} \neq 0$.

Step 3: Performing $R_3 \to R_3 - (a'_{32}/a'_{22})R_2$, we get

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a'_{22} & a'_{23} & b'_2 \\ 0 & 0 & a''_{33} & b''_3 \end{array}\right]$$

where

$$\begin{aligned} a''_{33} &= a'_{33} - (a'_{32}/a'_{22}) \times a'_{23} \\ b''_{33} &= b'_{33} - (a'_{32}/a'_{22}) \times b'_2 \end{aligned}$$

This is the end of the forward elimination. From the system of equations obtained from step 3, the values of x_1, x_2 and x_3 can be obtained by back substitution. This procedure is called partial pivoting. If this is impossible, then the matrix is singular and the system has no solution.

CAYLEY–HAMILTON THEOREM

According to the Cayley—Hamilton theorem, every square matrix satisfies its own characteristic equations. Hence, if

$$|A - \lambda I| = (-1)^n(\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n)$$

is the characteristic polynomial of a matrix A of order n, then the matrix equation

$$X^n + a_1X^{n-1} + a_2X^{n-2} + \cdots + a_nI = 0$$

is satisfied by $X = A$.

EIGENVALUES AND EIGENVECTORS

If $A = [a_{ij}]_{n\times n}$ is a square matrix of order n, then the vector equation $AX = \lambda X$, where $X = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix}$ is an unknown vector and λ is an unknown scalar value, is called an eigenvalue problem. To solve the problem, we need to determine the value of X's and λ's to satisfy the above-mentioned vector. Note that the zero vector (i.e. $X = 0$) is not of our interest. A value of λ for which the above equation has a solution $X \neq 0$ is called an *eigenvalue* or *characteristic value* of the matrix A. The corresponding solutions $X \neq 0$ of the equation are called the *eigenvectors* or *characteristic vectors* of A corresponding to that eigenvalue, λ. The set of all the eigenvalues of A is called the *spectrum* of A. The largest of the absolute values of the eigenvalues of A is called the *spectral radius* of A. The sum of the elements of the principal diagonal of a matrix A is called the *trace* of A.

Properties of Eigenvalues and Eigenvectors

Some of the main characteristics of eigenvalues and eigenvectors are discussed in the following points:

1. If $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ are the eigenvalues of A, then $k\lambda_1, k\lambda_2, k\lambda_3, \ldots, k\lambda_n$ are eigenvalues of kA, where k is a constant scalar quantity.
2. If $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ are the eigenvalues of A, then $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}, \ldots, \frac{1}{\lambda_n}$ are the eigenvalues of A^{-1}.
3. If $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ are the eigenvalues of A, then $\lambda_1^k, \lambda_2^k, \lambda_3^k, \ldots, \lambda_n^k$ are the eigenvalues of A^k.
4. If $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ are the eigenvalues of A, then $\frac{|A|}{\lambda_1}, \frac{|A|}{\lambda_2}, \frac{|A|}{\lambda_3}, \ldots, \frac{|A|}{\lambda_n}$ are the eigenvalues of adj A.
5. The eigenvalues of a matrix A are equal to the eigenvalues of A^T.
6. The maximum number of distinct eigenvalues is n, where n is the size of the matrix A.
7. The trace of a matrix is equal to the sum of the eigenvalues of a matrix.
8. The product of the eigenvalues of a matrix is equal to the determinant of that matrix.
9. If A and B are similar matrices, i.e. $A = IB$, then A and B have the same eigenvalues.
10. If A and B are two matrices of same order, then the matrices AB and BA have the same eigenvalues.
11. The eigenvalues of a triangular matrix are equal to the diagonal elements of the matrix.

SOLVED EXAMPLES

1. If $A = \begin{bmatrix} 1 & 6 \\ 2 & 4 \end{bmatrix}, B = \begin{bmatrix} 4 & -7 \\ 3 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 9 \\ -8 & 2 \end{bmatrix}$, then find the value of $3A + 2B - 4C$.

Solution: We have

$$A = \begin{bmatrix} 1 & 6 \\ 2 & 4 \end{bmatrix}, B = \begin{bmatrix} 4 & -7 \\ 3 & 5 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 9 \\ -8 & 2 \end{bmatrix}$$

Now,

$$3A = \begin{bmatrix} 3 & 18 \\ 6 & 12 \end{bmatrix} \quad (1)$$

$$2B = \begin{bmatrix} 8 & -14 \\ 6 & 10 \end{bmatrix} \quad (2)$$

$$4C = \begin{bmatrix} 4 & 36 \\ -32 & 8 \end{bmatrix} \quad (3)$$

Now calculating Eqs. (1) + (2) − (3), we get

$$3A + 2B - 4C = \begin{bmatrix} 3 & 18 \\ 6 & 12 \end{bmatrix} + \begin{bmatrix} 8 & -14 \\ 6 & 10 \end{bmatrix} - \begin{bmatrix} 4 & 36 \\ -32 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 3+8-4 & 18-14-36 \\ 6+6+32 & 12+10-8 \end{bmatrix} = \begin{bmatrix} 7 & -32 \\ 44 & 14 \end{bmatrix}$$

2. Find a, b, c and d such that $\begin{bmatrix} a-b & 2c+d \\ 2a-b & 2a+d \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 12 & 15 \end{bmatrix}$.

Solution: We know that the corresponding elements of two equal matrices are equal.

Therefore,

$$a - b = 5 \quad (1)$$
$$2a - b = 12 \quad (2)$$
$$2c + d = 3 \quad (3)$$
$$2a + d = 15 \quad (4)$$

Subtracting Eq. (1) from Eq. (2), we get

$$a = 7$$
$$\Rightarrow 7 - b = 5 \Rightarrow b = 2$$

Substituting the value of a in Eq. (4), we get

$$14 + d = 15 \Rightarrow d = 1$$

Substituting the value of d in Eq. (3), we get

$$2c + 1 = 3 \Rightarrow c = 1$$

Hence, $a = 7, b = 2, c = 1$ and $d = 1$.

3. If $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$, then find the value of A^2.

Solution: We have

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$$

Now,

$$A^2 = A \cdot A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4+0+1 & 0+0-1 & 2+0+0 \\ 4+2+3 & 0+1-3 & 2+3+0 \\ 2-2+0 & 0-1+0 & 1-3+0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix}$$

4. If $A = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & -1 & -4 \end{bmatrix}$, verify that $(AB)^T = B^T A^T$.

Solution: We have

$$A = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} -2 & -1 & -4 \end{bmatrix}$$

$$AB = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} -2 & -1 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 4 \\ -4 & -2 & -8 \\ -6 & -3 & -12 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 2 & -4 & -6 \\ 1 & -2 & -3 \\ 4 & -8 & -12 \end{bmatrix}$$

Also,

$$B^T A^T = \begin{bmatrix} -2 & -1 & -4 \end{bmatrix}^T \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}^T = \begin{bmatrix} -2 \\ -1 \\ -4 \end{bmatrix} \begin{bmatrix} -1 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -4 & -6 \\ 1 & -2 & -3 \\ 4 & -8 & -12 \end{bmatrix}$$

Hence, it can be seen that $(AB)^T = B^T A^T$.

5. For what value of x, the matrix $A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 2 & 1 \\ x & 2 & -3 \end{bmatrix}$ is singular?

Solution: The matrix A is singular if $\begin{vmatrix} 1 & -2 & 3 \\ 1 & 2 & 1 \\ x & 2 & -3 \end{vmatrix} = 0$

$$\Rightarrow 1(-6-2) - 1(6-6) + x(-2-6) = 0$$
$$\Rightarrow -8 - 8x = 0$$
$$\Rightarrow x = -1$$

6. Solve the following system of equations using Cramer's rule:

$$5x - 7y + z = 11$$
$$6x - 8y - z = 15$$
$$3x + 2y - 6z = 7$$

Solution: The given set of equations is

$$5x - 7y + z = 11$$
$$6x - 8y - z = 15$$
$$3x + 2y - 6z = 7$$

Now,

$$D = \begin{vmatrix} 5 & -7 & 1 \\ 6 & -8 & -1 \\ 3 & 2 & -6 \end{vmatrix}$$
$$= 5(48+2) + 7(-36+3) + 1(12+24)$$
$$= 250 - 231 + 36 = 55$$

$$D_1 = \begin{vmatrix} 11 & -7 & 1 \\ 15 & -8 & -1 \\ 7 & 2 & -6 \end{vmatrix}$$
$$= 11(48+2) + 7(-90+7) + 1(30+56)$$
$$= 550 - 581 + 86 = 55$$

$$D_2 = \begin{vmatrix} 5 & 11 & 1 \\ 6 & 15 & -1 \\ 3 & 7 & -6 \end{vmatrix}$$
$$= 5(-90+7) - 11(-36+3) + 1(42-45)$$
$$= -415 + 363 - 3 = -55$$

$$D_3 = \begin{vmatrix} 5 & -7 & 11 \\ 6 & -8 & 15 \\ 3 & 2 & 7 \end{vmatrix}$$
$$= 5(-56-30) + 7(42-45) + 11(12+24)$$
$$= -430 - 21 + 396 = -55$$

Now, using Cramer's rule,

$$x = \frac{D_1}{D} = \frac{55}{55} = 1,$$
$$y = \frac{D_2}{D} = \frac{-55}{55} = -1,$$
$$z = \frac{D_3}{D} = \frac{-55}{55} = -1$$

Hence, the solution of the given set of equations is $x = 1$, $y = -1$ and $z = -1$.

7. Find the adjoint of the matrix $A = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Solution: Let C_{ij} be the cofactor of A. Then

$$C_{11} = (-1)^{1+1}\begin{vmatrix} \cos\phi & 0 \\ 0 & 1 \end{vmatrix} = \cos\phi,\ C_{12} = (-1)^{1+2}\begin{vmatrix} \sin\phi & 0 \\ 0 & 1 \end{vmatrix}$$
$$= -\sin\phi,\ C_{13} = (-1)^{1+3}\begin{vmatrix} \sin\phi & \cos\phi \\ 0 & 0 \end{vmatrix} = 0$$
$$C_{21} = (-1)^{2+1}\begin{vmatrix} -\sin\phi & 0 \\ 0 & 1 \end{vmatrix} = \sin\phi,\ C_{22} = (-1)^{2+2}\begin{vmatrix} \cos\phi & 0 \\ 0 & 1 \end{vmatrix}$$
$$= \cos\phi,\ C_{23} = (-1)^{2+3}\begin{vmatrix} \cos\phi & -\sin\phi \\ 0 & 0 \end{vmatrix} = 0$$
$$C_{31} = (-1)^{3+1}\begin{vmatrix} -\sin\phi & 0 \\ \cos\phi & 0 \end{vmatrix} = 0,\ C_{32} = (-1)^{3+2}\begin{vmatrix} \cos\phi & 0 \\ \sin\phi & 0 \end{vmatrix}$$
$$= 0,\ C_{33} = (-1)^{3+3}\begin{vmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{vmatrix} = 1$$

Therefore,

$$\text{adj } A = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}^{T} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

8. If $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$, then what is the value of A^{-1}.

Solution: We have

$$A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$$

Now,

$$|A| = \begin{vmatrix} 3 & 2 \\ 7 & 5 \end{vmatrix} = 15 - 14 = 1 \neq 0$$

Hence, A is invertible.

The cofactors of A is given by

$$A_{11} = (-1)^{1+1} \cdot 5 = 5$$
$$A_{12} = (-1)^{1+2} \cdot 7 = -7$$
$$A_{21} = (-1)^{2+1} \cdot 2 = -2$$
$$A_{22} = (-1)^{2+2} \cdot 3 = 3$$

The adjoint of the matrix A is given by

$$\text{adj } A = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}^{T} = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$$

Hence, $A^{-1} = \dfrac{1}{1}\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$.

9. Show that the homogeneous system of equations has a non-trivial solution and find the solution.

$$x - 2y + z = 0$$
$$x + y - z = 0$$
$$3x + 6y - 5z = 0$$

Solution: The given system of equations can be written in the matrix form as follows:

$$\begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & -1 \\ 3 & 6 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which is similar to $AX = O$, where $A = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & -1 \\ 3 & 6 & -5 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Now,

$$|A| = \begin{vmatrix} 1 & -2 & 1 \\ 1 & 1 & -1 \\ 3 & 6 & -5 \end{vmatrix} = 1(-5+6) - 1(10-6)$$
$$+ 3(2-1) = 0$$

Thus, $|A| = 0$ and hence the given system of equations has a non-trivial solution.

Now, to find the solution, we put $z = k$ in the first two equations.

$$x - 2y = -k$$
$$x + y = k$$
$$\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -k \\ k \end{bmatrix}$$

which is similar to $AX = B$, where $A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} -k \\ k \end{bmatrix}$.

Now,

$$|A| = \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} = 3 \neq 0$$

Hence, A^{-1} exists.

Now, adj $A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$.

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$$

Now, $X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \dfrac{1}{3}\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -k \\ k \end{bmatrix} = \begin{bmatrix} k/3 \\ 2k/3 \end{bmatrix}$

$$\Rightarrow x = k/3, y = 2k/3$$

Hence, $x = k/3, y = 2k/3$ and $z = k$, where k is any real number that satisfies the given set of equations.

10. Check the following system of equations for consistency.

$$4x - 2y = 3$$
$$6x - 3y = 5$$

Solution: The given system of equations can be written as

$AX = B$, where $A = \begin{bmatrix} 4 & -2 \\ 6 & -3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

Now, $|A| = \begin{vmatrix} 4 & -2 \\ 6 & -3 \end{vmatrix} = -12 + 12 = 0$

So, the given system of equations is inconsistent or it has infinitely many solutions according to $(\text{adj} A)B \neq 0$ or $(\text{adj} A)B = 0$, respectively.

The cofactors can be calculated as follows:

$$C_{11} = (-1)^{1+1}(-3) = -3$$
$$C_{12} = (-1)^{1+2}(6) = -6$$
$$C_{21} = (-1)^{2+1}(-2) = 2$$
$$C_{22} = (-1)^{2+2}(4) = 4$$
$$\text{adj} A = \begin{bmatrix} -3 & -6 \\ 2 & 4 \end{bmatrix}^T = \begin{bmatrix} -3 & 2 \\ -6 & 4 \end{bmatrix}$$

Thus,

$$(\text{adj} A)B = \begin{bmatrix} -3 & 2 \\ -6 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} -9 + 10 \\ -18 + 20 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \neq 0$$

Hence, the given system of equations is inconsistent.

PRACTICE EXERCISE

1. What is the rank of the matrix $A = \begin{bmatrix} 1 & 3 & 5 & 1 \\ 2 & 4 & 8 & 0 \\ 3 & 1 & 7 & 5 \end{bmatrix}$?

(a) 1 (b) 2
(c) 3 (d) 4

2. What is the rank of the matrix $A = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$?

(a) 1 (b) 2
(c) 3 (d) 4

3. What is the rank of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$?

(a) 1 (b) 2
(c) 3 (d) 4

4. What is the rank of the matrix $A = \begin{bmatrix} 4 & 2 & 3 \\ 1 & 0 & 0 \\ 4 & 0 & 3 \end{bmatrix}$?

(a) 0 (b) 1
(c) 2 (d) 3

5. If $\begin{bmatrix} a+b & 3 \\ 5 & ab \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 5 & 3 \end{bmatrix}$, then what are the values of a and b?

(a) (2, 1) or (1, 2) (b) (2, 4) or (4, 2)
(c) (0, 3) or (3, 0) (d) (1, 3) or (3, 1)

6. If $A = \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 \\ 7 & 1 \end{bmatrix}$, then what is the value of $4A - 3B$?

(a) $\begin{bmatrix} -5 & -20 \\ -9 & 17 \end{bmatrix}$ (b) $\begin{bmatrix} 5 & 20 \\ 9 & -17 \end{bmatrix}$

(c) $\begin{bmatrix} -2 & -2 \\ -4 & 4 \end{bmatrix}$ (d) $\begin{bmatrix} -5 & 20 \\ 9 & 17 \end{bmatrix}$

7. What is the value of

$$\sin\theta \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix} + \cos\theta \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}?$$

(a) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} \sin\theta\cos\theta & \sin\theta + \cos\theta \\ \sin\theta - \cos\theta & \sin\theta\cos\theta \end{bmatrix}$

(d) $\begin{bmatrix} \sin\theta\cos\theta & 0 \\ 0 & \sin\theta + \cos\theta \end{bmatrix}$

8. If $B = \begin{bmatrix} 1 & 7 \\ 3 & 1 \end{bmatrix}, C = \begin{bmatrix} 4 & 1 \\ 6 & 8 \end{bmatrix}$, and $2A + 3B - 6C = 0$, then what is the value of A?

(a) $\begin{bmatrix} 21/2 & 27/2 \\ -15/2 & 45/2 \end{bmatrix}$ (b) $\begin{bmatrix} 21/4 & 27/4 \\ -15/4 & 45/4 \end{bmatrix}$

(c) $\begin{bmatrix} 21/2 & -15/2 \\ 27/2 & 45/2 \end{bmatrix}$ (d) $\begin{bmatrix} 21/4 & -15/4 \\ 27/4 & 45/4 \end{bmatrix}$

9. Find A and B, if

$$A+B=\begin{bmatrix}8 & 5\\8 & 13\end{bmatrix} \text{ and } A-B=\begin{bmatrix}6 & 1\\2 & 3\end{bmatrix}.$$

(a) $A=\begin{bmatrix}7 & 3\\5 & 8\end{bmatrix}, B=\begin{bmatrix}1 & 2\\3 & 5\end{bmatrix}$

(b) $A=\begin{bmatrix}1 & 2\\3 & 5\end{bmatrix}, B=\begin{bmatrix}7 & 3\\5 & 8\end{bmatrix}$

(c) $A=\begin{bmatrix}5 & 2\\5 & 6\end{bmatrix}, B=\begin{bmatrix}3 & 3\\3 & 7\end{bmatrix}$

(d) $A=\begin{bmatrix}10 & 5\\5 & 8\end{bmatrix}, B=\begin{bmatrix}4 & 4\\3 & 5\end{bmatrix}$

10. For what values of λ, the given set of equations has a unique solution?

$$2x+3y+5z=9$$
$$7x+3y-2z=8$$
$$2x+3y+\lambda z=9$$

(a) $\lambda = 15$
(b) $\lambda = 5$
(c) For all values except $\lambda = 15$
(d) For all values except $\lambda = 5$

11. For what values of λ, the given set of equations has a unique solution?

$$x+y+z=3$$
$$x+2y+3z=4$$
$$x+4y+\lambda z=6$$

(a) 5 (b) 7
(c) 9 (d) 0

12. How many solutions does the following system of equations have?

$$x+2y+z=6$$
$$2x+y+2z=6$$
$$x+y+z=5$$

(a) One solution
(b) Infinite solutions
(c) No solutions
(d) None of the above

13. If $A=\begin{bmatrix}1 & 2 & -7\\3 & 1 & 5\\4 & 7 & 1\end{bmatrix}$ and $B=\begin{bmatrix}3 & -5 & 1\\4 & 8 & 5\\1 & 2 & 6\end{bmatrix}$, then what is the value of $(A \times B)$?

(a) $\begin{bmatrix}4 & -3 & -31\\18 & 3 & 38\\41 & 38 & 45\end{bmatrix}$ (b) $\begin{bmatrix}4 & -3 & -38\\-3 & 3 & 38\\-31 & 31 & 45\end{bmatrix}$

(c) $\begin{bmatrix}11 & -3 & -31\\18 & 9 & 18\\41 & 38 & 35\end{bmatrix}$ (d) $\begin{bmatrix}4 & 3 & -31\\18 & -3 & 38\\45 & 38 & 41\end{bmatrix}$

14. If $A=\begin{bmatrix}k & 0\\1 & 4\end{bmatrix}$ and $B=\begin{bmatrix}4 & 0\\6 & 16\end{bmatrix}$, then what is the value of k for which $A^2 = B$?

(a) −1 (b) −2
(c) 1 (d) 2

15. If $A=\begin{bmatrix}1 & 0\\-1 & 7\end{bmatrix}$ and $I=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$, then what is the value of k for which $A^2 = 8A + kI$?

(a) 7 (b) −7
(c) 10 (d) 8

16. If $\begin{bmatrix}2 & -1\\1 & 0\\-3 & 4\end{bmatrix}A=\begin{bmatrix}-1 & -8 & -10\\1 & -2 & -5\\9 & 22 & 15\end{bmatrix}$, then what is the value of A?

(a) $\begin{bmatrix}3 & 4 & 0\\1 & 3 & 4\end{bmatrix}$ (b) $\begin{bmatrix}3 & 4 & 0\\1 & -2 & -5\end{bmatrix}$

(c) $\begin{bmatrix}1 & -2 & -5\\3 & 4 & 0\end{bmatrix}$ (d) $\begin{bmatrix}1 & 3 & 4\\3 & 4 & 0\end{bmatrix}$

17. If $A=\begin{bmatrix}1 & 2 & 2\\2 & 1 & -2\\a & 2 & b\end{bmatrix}$ is a matrix such that $AA^T = 9I_3$, then what are the values of a and b?

(a) $a = -1, b = -2$
(b) $a = -2, b = -1$
(c) $a = 1, b = 2$
(d) $a = 2, b = 1$

18. If $A=\begin{bmatrix}8 & 4 & 6\\4 & 0 & 2\\x & 6 & 0\end{bmatrix}$ is singular, then what is the value of x?

(a) 12 (b) 8
(c) 4 (d) 1

19. If $A=\begin{bmatrix}2 & 3\\5 & -2\end{bmatrix}$, then what is the value of A^{-1}?

(a) $\frac{1}{19}\begin{bmatrix}-2 & -3\\-5 & 2\end{bmatrix}$ (b) $\frac{1}{29}\begin{bmatrix}-2 & -3\\-5 & 2\end{bmatrix}$

(c) $\frac{1}{19}\begin{bmatrix}2 & 3\\5 & -2\end{bmatrix}$ (d) $\frac{1}{29}\begin{bmatrix}2 & 3\\5 & -2\end{bmatrix}$

20. What is the value of I^T, where I is an identity matrix of order 3?

(a) $\begin{bmatrix}1 & 0 & 0\\1 & 0 & 0\\1 & 0 & 0\end{bmatrix}$ (b) $\begin{bmatrix}0 & 0 & 1\\0 & 1 & 0\\1 & 0 & 0\end{bmatrix}$

(c) $\begin{bmatrix}1 & 1 & 1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$ (d) $\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$

21. If $A = \begin{bmatrix} 1 & 7 & -1 \\ 3 & 2 & 2 \\ 4 & 5 & 1 \end{bmatrix}$, then what is the first row of A^T?

(a) [1 7 −1] (b) [1 3 4]
(c) [3 2 2] (d) [4 5 1]

22. If $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$, then what is the value of A^{-1}?

(a) $\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 7 & -1 & -1 \\ -3 & 0 & 1 \\ -3 & 1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} -3 & -3 & 7 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

23. Calculate the adjoint of the matrix $A = \begin{bmatrix} 8 & 4 & 2 \\ 2 & 9 & 4 \\ 1 & 2 & 8 \end{bmatrix}$.

(a) $\begin{bmatrix} 64 & 28 & 2 \\ 12 & 62 & 18 \\ 6 & 36 & -64 \end{bmatrix}$ (b) $\begin{bmatrix} 62 & -28 & -2 \\ -12 & 62 & -38 \\ -2 & -12 & -62 \end{bmatrix}$

(c) $\begin{bmatrix} 64 & -28 & -2 \\ -12 & 62 & -28 \\ -5 & -12 & 64 \end{bmatrix}$ (d) $\begin{bmatrix} 64 & 28 & 24 \\ 10 & 62 & 48 \\ 6 & 36 & -64 \end{bmatrix}$

24. If $A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$, then what is the value of B such that $AB = C$?

(a) $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 4 \\ 1 & 3 \end{bmatrix}$

25. If $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}$ and if $ABC = I_2$, then what is the value of C?

(a) $\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

26. What is the value of $(AB)^{-1}B$?

(a) A^{-1} (b) B
(c) A (d) $A^{-1}B^{-1}$

27. If $A = \begin{bmatrix} x & 2 & 0 \\ 2 & 0 & 1 \\ 6 & 3 & 0 \end{bmatrix}$ is singular, then what is the value of x?

(a) 0 (b) 2
(c) 4 (d) 6

28. What is the nullity of the matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}$?

(a) 0 (b) 1
(c) 2 (d) 3

29. What are the eigenvalues of $A = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$?

(a) 1, 4 (b) 2, 3
(c) 0, 5 (d) 1, 5

30. What are the eigenvalues of $A = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$?

(a) 1, 4, 4 (b) 1, 4, − 4
(c) 3, 3, 3 (d) 1, 2, 6

31. What is the eigenvector of the matrix $A = \begin{bmatrix} 5 & -4 \\ -1 & 2 \end{bmatrix}$?

(a) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (b) $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$

(c) $\begin{bmatrix} -2 \\ -1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

32. What are the eigenvectors of the matrix $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$?

(a) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ (b) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

33. What are the eigenvalues of the matrix $A = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$?

(a) $\sqrt{2}, -\sqrt{2}, 1$ (b) $i, -i, 1$

(c) $2, -2, 1$ (d) $0, \frac{1}{2}, \frac{1}{2}$

34. What are the eigenvalues of the matrix $A = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$?

(a) 1, 2, 3 (b) 4, 4, 5
(c) 3, 5, 6 (d) 3, 3, 7

35. Which one of the following options is not the eigenvector of matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$?

(a) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

36. What are the eigenvectors of the matrix $A = \begin{bmatrix} 2 & 5 & 0 \\ 0 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix}$?

(a) $\begin{bmatrix} k \\ 0 \\ k \end{bmatrix}, \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ k \\ 2k \end{bmatrix}$ (b) $\begin{bmatrix} 0 \\ 0 \\ k \end{bmatrix}, \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ k \\ 2k \end{bmatrix}$

(c) $\begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2k \end{bmatrix}$ (d) $\begin{bmatrix} 0 \\ 0 \\ k \end{bmatrix}, \begin{bmatrix} k \\ k \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2k \\ k \end{bmatrix}$

37. What is the sum of eigenvalues of $A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 5 & 4 \\ 1 & 6 & 2 \end{bmatrix}$?

(a) 8 (b) 10
(c) 4 (d) 5

38. What is the value of x and y if $A = \begin{bmatrix} x & y \\ -4 & 10 \end{bmatrix}$ and eigenvalues of A are 4 and 8?

(a) $x = 3, y = 2$ (b) $x = 2, y = 4$
(c) $x = 4, y = 2$ (d) $x = 2, y = 3$

39. What are the eigenvalues of $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$?

(a) $1, -1$ (b) $1, i$
(c) $i, -i$ (d) $0, 1$

40. What are the values of x and y, if $A = \begin{bmatrix} 3 & 0 & 0 \\ 5 & x & 0 \\ 3 & 6 & y \end{bmatrix}$ and eigenvalues of A are 3, 4 and 1?

(a) $x = 2, y = 2$ (b) $x = 3, y = 2$
(c) $x = 2, y = 3$ (d) $x = 4, y = 1$

ANSWERS

1. (c)	**8.** (c)	**15.** (b)	**22.** (a)	**29.** (c)	**36.** (b)
2. (b)	**9.** (a)	**16.** (c)	**23.** (c)	**30.** (a)	**37.** (a)
3. (b)	**10.** (d)	**17.** (b)	**24.** (b)	**31.** (b)	**38.** (d)
4. (d)	**11.** (b)	**18.** (a)	**25.** (d)	**32.** (c)	**39.** (c)
5. (d)	**12.** (c)	**19.** (c)	**26.** (a)	**33.** (a)	**40.** (d)
6. (a)	**13.** (a)	**20.** (d)	**27.** (c)	**34.** (b)	
7. (b)	**14.** (d)	**21.** (b)	**28.** (b)	**35.** (d)	

EXPLANATIONS AND HINTS

1. (c) Matrix $A = \begin{vmatrix} 1 & 3 & 5 & 1 \\ 2 & 4 & 8 & 0 \\ 3 & 1 & 7 & 5 \end{vmatrix}$

Maximum possible rank = 3

Now, consider 3 × 3 minors

$$\begin{vmatrix} 1 & 3 & 5 \\ 2 & 4 & 8 \\ 3 & 1 & 7 \end{vmatrix} = (28-8) - 2\ (21-5) + 3\ (24-20)$$

$$= 20 - 32 + 12 = 0$$

$$\begin{vmatrix} 3 & 5 & 1 \\ 4 & 8 & 0 \\ 1 & 7 & 5 \end{vmatrix} = 3(40 - 0) - 4(25 - 7) + 1(0 - 8)$$

$$= 120 - 72 - 8 = 40 \neq 0$$

Hence, rank of $A = 3$.

2. (b) Matrix $A = \begin{vmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{vmatrix}$

Maximum possible rank = 3

Now, consider 3 × 3 minors

$$\begin{vmatrix} 4 & 2 & 1 \\ 6 & 3 & 4 \\ 2 & 1 & 0 \end{vmatrix} = 0$$

$$\begin{vmatrix} 2 & 1 & 3 \\ 3 & 4 & 7 \\ 1 & 0 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 4 & 1 & 3 \\ 6 & 3 & 7 \\ 2 & 0 & 1 \end{vmatrix} = 0$$

and $$\begin{vmatrix} 4 & 2 & 3 \\ 6 & 3 & 7 \\ 2 & 1 & 1 \end{vmatrix} = 0$$

Now, because all 3 × 3 minors are zero, let us consider 2 × 2 minors

$$\begin{vmatrix} 4 & 2 \\ 6 & 3 \end{vmatrix} = 0$$

$$\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 8 - 3 = 5 \neq 0$$

Hence, rank of A is 2.

3. (b) Matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

Maximum rank = 3

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = (-1-0) - 1(1-1) + 1(0+1)$$

$$= -1 + 1 = 0$$

Hence, rank ≠ 3.

Now, let us consider 2 × 2 minors

$$\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = (-1-1) = -2 \neq 0$$

Hence, rank of A is 2.

4. (d) Matrix $A = \begin{bmatrix} 4 & 2 & 3 \\ 1 & 0 & 0 \\ 4 & 0 & 3 \end{bmatrix}$

Maximum rank = 3

$$\begin{bmatrix} 4 & 2 & 3 \\ 1 & 0 & 0 \\ 4 & 0 & 3 \end{bmatrix} = 4(0) - 1(6-0) + 4(0) = -6 \neq 0$$

Hence, rank of $A = 3$.

5. (d) Since both the matrices are equal

$$\begin{bmatrix} a+b & 3 \\ 5 & ab \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 5 & 3 \end{bmatrix}$$

$$\Rightarrow a + b = 4 \quad (1)$$

$$ab = 3 \quad (2)$$

From Eq. (1), we have

$$a = 4 - b$$

Substituting the value of a in Eq. (2), we get

$$(4 - b)\ b = 3$$

$$\Rightarrow 4b - b^2 = 3$$

$$\Rightarrow b^2 - 4b + 3 = 0$$

$$\Rightarrow (b - 3)\ (b - 1) = 0$$

$$\Rightarrow b = 3, 1$$

For values of b, $a = 4 - 3, 4 - 1 = 1, 3$

Therefore, the values of a and b = (1, 3) or (3, 1)

6. (a) $A = \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 4 \\ 7 & 1 \end{bmatrix}$

$$4A = \begin{bmatrix} 4 & -8 \\ 12 & 20 \end{bmatrix}, \quad 3B = \begin{bmatrix} 9 & 12 \\ 21 & 3 \end{bmatrix}$$

$$4A - 3B = \begin{bmatrix} 4-9 & -8-12 \\ 12-21 & 20-3 \end{bmatrix} = \begin{bmatrix} -5 & -20 \\ -9 & 17 \end{bmatrix}$$

7. (b) We have

$$\cos\theta \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ -\sin\theta\cos\theta & \cos^2\theta \end{bmatrix} + \begin{bmatrix} \sin^2\theta & -\sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \sin\theta\cos\theta - \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \sin\theta\cos\theta & \cos^2\theta + \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

8. (c) We have

$$2A + 3B - 6C = 0$$

$$A = \frac{1}{2}(6C - 3B)$$

Also, $B = \begin{bmatrix} 1 & 7 \\ 3 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 4 & 1 \\ 6 & 8 \end{bmatrix}$

$$A = \frac{1}{2}\left(6\begin{bmatrix} 4 & 1 \\ 6 & 8 \end{bmatrix} - 3\begin{bmatrix} 1 & 7 \\ 3 & 1 \end{bmatrix}\right)$$

$$= \frac{1}{2}\left(\begin{bmatrix} 24 & 6 \\ 36 & 48 \end{bmatrix} - \begin{bmatrix} 3 & 21 \\ 9 & 3 \end{bmatrix}\right)$$

$$= \frac{1}{2}\left(\begin{bmatrix} 21 & -15 \\ 27 & 45 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 21/2 & -15/2 \\ 27/2 & 45/2 \end{bmatrix}$$

9. (a) We have

$$A + B = \begin{bmatrix} 8 & 5 \\ 8 & 13 \end{bmatrix} \text{ and } A - B = \begin{bmatrix} 6 & 1 \\ 2 & 3 \end{bmatrix}$$

$$\therefore (A + B) + (A - B) = \begin{bmatrix} 8 & 5 \\ 8 & 13 \end{bmatrix} + \begin{bmatrix} 6 & 1 \\ 2 & 3 \end{bmatrix}$$

$$2A = \begin{bmatrix} 14 & 6 \\ 10 & 16 \end{bmatrix} \Rightarrow A = \frac{1}{2}\begin{bmatrix} 14 & 6 \\ 10 & 16 \end{bmatrix} = \begin{bmatrix} 7 & 3 \\ 5 & 8 \end{bmatrix}$$

$$\text{Also, } (A + B) - (A - B) = \begin{bmatrix} 8 & 5 \\ 8 & 13 \end{bmatrix} - \begin{bmatrix} 6 & 1 \\ 2 & 3 \end{bmatrix}$$

$$2B = \begin{bmatrix} 2 & 4 \\ 6 & 10 \end{bmatrix} \Rightarrow B = \frac{1}{2}\begin{bmatrix} 2 & 4 \\ 6 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$$

Thus, $A = \begin{bmatrix} 7 & 3 \\ 5 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$.

10. (d) The given set of equations can be written as

$$\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ 9 \end{bmatrix}$$

The system will have a unique solution if the rank of coefficient matrix is 3.

Thus,

$$\begin{vmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{vmatrix} = 2(3\lambda + 6) - 7(3\lambda - 15) + 2(-6 - 15)$$

$$= 6\lambda + 12 - 21\lambda + 105 - 12 - 30$$

$$= -15\lambda + 75$$

$$= 15(5 - \lambda)$$

For rank = 3,

$$\begin{vmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{vmatrix} \neq 0$$

$$\therefore 15(5 - \lambda) \neq 0$$

$$\lambda \neq 5$$

11. (b) The given set of equations can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & k \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

$$\text{Thus, } \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & k \end{vmatrix} = 1(2k - 12) - 1(k - 4) + 1(3 - 2)$$

$$= 2k - 12 - k + 4 + 1$$

$$= k - 7$$

Now, for a system to be unique

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & k \end{vmatrix} \neq 0$$

$$\Rightarrow k - 7 \neq 0$$

$$\Rightarrow k \neq 7$$

Thus, the value of k for which the given set of equations does not have a unique solution is 7.

12. (c) The given system can be written as

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 5 \end{bmatrix}$$

$$\text{Now, } \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{vmatrix} = 1(1 - 2) - 2\ (2 - 1) + 1(4 - 1)$$

$$= -1 - 2 + 3 = 0$$

Hence, rank of matrix is not 3.

Now, taking a minor from the matrix

$$\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 1 - 4 = -3 \neq 0$$

Hence, rank of matrix = 2.

Now, rank of matrix is less than the number of variables.

Hence, the system is inconsistent or has no solution.

13. (a) We have

$$A = \begin{bmatrix} 1 & 2 & -7 \\ 3 & 1 & 5 \\ 4 & 7 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & -5 & 1 \\ 4 & 8 & 5 \\ 1 & 2 & 6 \end{bmatrix}$$

$$A \times B = \begin{bmatrix} 3 + 8 + (-7) & -5 + 16 - 14 & 1 + 10 - 42 \\ 9 + 4 + 5 & -15 + 8 + 10 & 3 + 5 + 30 \\ 12 + 28 + 1 & -20 + 56 + 2 & 4 + 35 + 6 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -3 & -31 \\ 18 & 3 & 38 \\ 41 & 38 & 45 \end{bmatrix}$$

14. (d) We have

$$A = \begin{bmatrix} k & 0 \\ 1 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & 0 \\ 6 & 16 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} k & 0 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} k & 0 \\ 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} k^2+0 & 0+0 \\ k+4 & 0+16 \end{bmatrix} = \begin{bmatrix} k^2 & 0 \\ k+4 & 16 \end{bmatrix}$$

Now, since $A^2 = B$

$$\begin{bmatrix} k^2 & 0 \\ k+4 & 16 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 6 & 16 \end{bmatrix}$$

$$k^2 = 4 \text{ and } k + 4 = 6$$

$$k = \pm 2 \text{ and } k = 2$$

Hence, value of $k = 2$.

15. (b) We have

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix} \quad \text{and} \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^2 \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1+0 & 0 \\ -1-7 & 0+49 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -8 & 49 \end{bmatrix}$$

$$= 8A = 8\begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ -8 & 56 \end{bmatrix}$$

$$kI = k\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

Now,

$$A^2 = 8A + kI$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ -8 & 49 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ -8 & 56 \end{bmatrix} + \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ -8 & 49 \end{bmatrix} = \begin{bmatrix} 8+k & 0 \\ -8 & 56+k \end{bmatrix}$$

or $\quad 8 + k = 1 \quad \text{and} \quad 56 + k = 49$

$$\Rightarrow k = -7$$

16. (c) As product is a 3×3 matrix and one of the matrix is 3×2, the order of A is 2×3.

Consider $A = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix}$, then

$$\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix} = \begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 - x_2 & 2y_1 - y_2 & 2z_1 - z_2 \\ x_1 & y_1 & z_1 \\ -3x_1 + 4x_2 & -3y_1 + 4y_2 & -3z_1 + 4z_2 \end{bmatrix} = \begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$$

Now, $2x_1 - x_2 = -1$, $x_1 = 1$

$$\Rightarrow 2 - x_2 = -1$$

$$\Rightarrow x_2 = 3$$

Also, $2y_1 - y_2 = -8$, $y_1 = -2$

$$\Rightarrow -4 - y_2 = -8$$

$$\Rightarrow y_2 = 4$$

Also, $2z_1 - z_2 = -10$, $z_1 = -5$

$$\Rightarrow z_2 = 0$$

Thus, $A = \begin{bmatrix} 1 & -2 & -5 \\ 3 & 4 & 0 \end{bmatrix}$.

17. (b) We have

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix}$$

$$AA^T = 9I_3$$

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}\begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix} = 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 9 & 0 & a+2b+4 \\ 0 & 9 & 2a+2-2b \\ a+2b+4 & 2a+2-2b & a^2+4+b^2 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$a + 2b + 4 = 0 \quad \Rightarrow a + 2b = -4$$

$$2a + 2 - 2b = 0 \quad \Rightarrow a - b = -1$$

Solving above equations,

$$3b = -3 \Rightarrow b = -1$$

$$a = -4 + 2 = -2$$

Hence, $a = -2$ and $b = -1$.

18. (a) We know that $A = \begin{bmatrix} 8 & 4 & 6 \\ 4 & 0 & 2 \\ x & 6 & 0 \end{bmatrix}$ is singular.

Hence, $\begin{vmatrix} 8 & 4 & 0 \\ 4 & 0 & 2 \\ x & 6 & 0 \end{vmatrix} = 0$

$$8(0 - 12) - 4(0) + x(8 - 0) = 0$$

$$-96 + 8x = 0$$

$$8x = 96$$

$$x = 12$$

19. (c) We have

$$A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$$

$$|A| = \{[2 \times (-2)] - (3 \times 5)\} = -4 - 15 = -19 \neq 0$$

Hence, A is invertible.

Now, cofactors of the matrix A are

$$C_{11} = -2$$

$$C_{12} = -5$$

$$C_{21} = -3$$

$$C_{22} = 2$$

$$\text{adj } A = \begin{bmatrix} -2 & -5 \\ -3 & 2 \end{bmatrix}^T = \begin{bmatrix} -2 & -3 \\ -5 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj A} = \frac{-1}{19}\begin{bmatrix} -2 & -3 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 2/19 & 3/19 \\ 5/19 & -2/19 \end{bmatrix}$$

20. (d) $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad I^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

21. (b) $A = \begin{bmatrix} 1 & 7 & -1 \\ 3 & 2 & 2 \\ 4 & 5 & 1 \end{bmatrix}$

$$A^T = \begin{bmatrix} 1 & 3 & 4 \\ 7 & 2 & 5 \\ -1 & 2 & 1 \end{bmatrix}$$

The first row of transport of $A = [1 \quad 3 \quad 4]$.

22. (a) $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

$$|A| = 1(16 - 9) - 1(12 - 9) + 1(9 - 12)$$
$$= 7 - 3 - 3 = 1 \neq 0$$

Hence, A is invertible.

Now, cofactors of the matrix A are given as

$$C_{11} = (-1)^{1+1}\begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix} = 7$$

$$C_{12} = (1)^{1+2}\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = -1$$

$$C_{13} = (-1)^{1+3}\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} = -1$$

$$C_{21} = (-1)^{2+1}\begin{vmatrix} 3 & 3 \\ 3 & 4 \end{vmatrix} = -3$$

$$C_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 1$$

$$C_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 0$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix} 3 & 3 \\ 4 & 3 \end{vmatrix} = -3$$

$$C_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 0$$

$$C_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 1$$

Thus, $\text{adj } A = \begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Hence, $A^{-1} = \frac{1}{|A|}\text{adj} A$

$$= \frac{1}{1}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

23. (c) We have

$$A = \begin{bmatrix} 8 & 4 & 2 \\ 2 & 9 & 4 \\ 1 & 2 & 8 \end{bmatrix}$$

Now, cofactors of the matrix A are given as

$$C_{11} = \begin{vmatrix} 9 & 4 \\ 2 & 8 \end{vmatrix}(-1)^{1+1} = 64$$

$$C_{12} = \begin{vmatrix} 2 & 4 \\ 1 & 8 \end{vmatrix}(-1)^{1+2} = -12$$

$$C_{13} = \begin{vmatrix} 2 & 9 \\ 1 & 2 \end{vmatrix}(-1)^{1+3} = -5$$

$$C_{21} = \begin{vmatrix} 4 & 2 \\ 2 & 8 \end{vmatrix}(-1)^{2+1} = -28$$

$$C_{22} = \begin{vmatrix} 8 & 2 \\ 1 & 8 \end{vmatrix}(-1)^{2+2} = 62$$

$$C_{23} = \begin{vmatrix} 8 & 4 \\ 1 & 2 \end{vmatrix}(-1)^{2+3} = -12$$

$$C_{31} = \begin{vmatrix} 4 & 2 \\ 9 & 4 \end{vmatrix}(-1)^{3+1} = -2$$

$$C_{32} = \begin{vmatrix} 8 & 2 \\ 2 & 4 \end{vmatrix}(-1)^{3+2} = -28$$

$$C_{33} = \begin{vmatrix} 8 & 4 \\ 2 & 9 \end{vmatrix}(-1)^{3+3} = 64$$

Thus,

$$\text{adj } A = \begin{bmatrix} 64 & -12 & -5 \\ -28 & 62 & -12 \\ -2 & -28 & 64 \end{bmatrix}^T = \begin{bmatrix} 64 & -28 & -2 \\ -12 & 62 & -28 \\ -5 & -12 & 64 \end{bmatrix}$$

24. (b) We know

$$A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$

Now, $\quad AB = C$

$$B = CA^{-1}$$

Now, we can calculate A^{-1} as follows:

$$|A| = 4 + 2 = 6 \neq 0$$

Hence, A is invertible. Now, cofactor of the matrix A are given as

$$C_{11} = (-1)^{1+1}\ (1) = 1$$
$$C_{12} = (-1)^{2+1}\ (-1) = 1$$
$$C_{21} = (-1)^{2+1}\ (2) = -2$$
$$C_{22} = (-1)^{2+2}\ (4) = 4$$

$$\text{adj} A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{6}\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} \times \frac{1}{6}\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$$

25. (d) We know that

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}$$

Now, $|A| = (4 - 3) = 1 \neq 0$

$$|B| = (9 - 10) = -1 \neq 0$$

Hence, both A and B are invertible.

Now, $\quad ABC = I$

$$C = A^{-1}B^{-1}I$$

or $\quad C = A^{-1}B^{-1}$

Calculating cofactors of A,

$$C_{A_{11}} = (-1)^{1+1}(2) = 2$$

$$C_{A_{12}} = (-1)^{1+2}(3) = -3$$

$$C_{A_{21}} = (-1)^{2+1}(1) = -1$$

$$C_{A_{22}} = (-1)^{2+2}(2) = 2$$

$$\text{adj}\,A = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}^T = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{1}\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

Calculating cofactors of B,

$$C_{B_{11}} = (-1)^{1+1}(-3) = -3$$

$$C_{B_{12}} = (-1)^{1+2}(5) = -5$$

$$C_{B_{21}} = (-1)^{2+1}(2) = -2$$

$$C_{B_{22}} = (-1)^{2+2}(-3) = -3$$

$$\text{adj}\,B = \begin{bmatrix} -3 & -5 \\ -2 & -3 \end{bmatrix}^T = \begin{bmatrix} -3 & -2 \\ -5 & -3 \end{bmatrix}$$

$$B^{-1} = \frac{1}{|B|}\text{adj}\,B = \frac{1}{(-1)}\begin{bmatrix} -3 & -2 \\ -5 & -3 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$$

Now,

$$C = A^{-1}B^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 6-5 & -9+10 \\ 4-3 & -6+6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

26. (a) $(AB)^{-1} = (A^{-1}B^{-1})\ B$

$$= A^{-1}[(B^{-1})\cdot B]$$

$$= A^{-1}I = A^{-1}$$

27. (c) We know

$$A = \begin{bmatrix} x & 2 & 0 \\ 2 & 0 & 1 \\ 6 & 3 & 0 \end{bmatrix}$$

For A to be singular,

$$|A| = 0$$

$$x(0-3) - 2(0-0) + 6(2) = 0$$

$$-3x + 12 = 0 \Rightarrow x = \frac{-12}{-3}$$

$$\Rightarrow x = 4$$

28. (b) We know

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}$$

$$|A| = 0(1-0) - 1(-1-0) + (-1)\ (0+1)$$

$$= 0 + 1 - 1 = 0$$

Hence, rank is not 3.

Choosing a minor from A, we get

$$\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = 0 - 1 = -1 \neq 0$$

Therefore, rank $(A) = 2$

Now, nullity = Number of columns − Rank of matrix

$$= 3 - 2$$

$$= 1$$

29. (c) We know

$$A = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$$

Now,

$$[A - \lambda I] = \begin{vmatrix} 4-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)(1-\lambda) - (-2)(-2) = 0$$

$$\lambda^2 - 5\lambda + 4 - 4 = 0$$

$$\lambda(\lambda - 5) = 0$$

$$\lambda = 0,\ 5$$

Hence, eigenvalues are 0 and 5.

30. (a) We know

$$A=\begin{bmatrix}3 & -1 & -1\\ -1 & 3 & -1\\ -1 & -1 & 3\end{bmatrix}$$

Now,

$$|A-\lambda I|=\begin{vmatrix}3-\lambda & -1 & -1\\ -1 & 3-\lambda & -1\\ -1 & -1 & 3-\lambda\end{vmatrix}=0$$

$$(3-\lambda)\left[(3-\lambda)^2-1\right]-(-1)[-(3-\lambda)-1]+(-1)[1+(3-\lambda)]=0$$

$$[(3-\lambda)+1]\{(3-\lambda)[(3-\lambda)-1]-(1)-1\}=0$$

$$(4-\lambda)\left[\lambda^2-5\lambda+4\right]=0\Rightarrow(4-\lambda)(\lambda-1)(\lambda-4)=0$$

$$\lambda=1,\,4,\,4$$

31. (b) We have

$$A=\begin{bmatrix}5 & -4\\ -1 & 2\end{bmatrix}$$

The characteristic equation is given by

$$|A-\lambda I|=0$$

$$\begin{vmatrix}5-\lambda & -4\\ -1 & 2-\lambda\end{vmatrix}=0$$

$$(5-\lambda)(2-\lambda)-4=0$$

$$\lambda^2-7\lambda+6=0$$

$$(\lambda-6)(\lambda-1)=0$$

$$\lambda=6,\,1$$

$\therefore$ Eigenvalues of $A=1,\,6$

Now, using $|A-\lambda|\widehat{X}=0$

and substituting $\lambda=1$, we get

$$\begin{bmatrix}4 & -4\\ -1 & 1\end{bmatrix}\begin{bmatrix}X_1\\ X_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$4X_1-4X_2=0$$

$$-X_1+X_2=0$$

or $\quad X_1=X_2$

Now, the solution is $X_1=X_2=k$.

Hence, from the given options, the solution is $\begin{bmatrix}2\\ 2\end{bmatrix}$.

32. (c) We have

$$A=\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

The characteristic equation is given by

$$|A-\lambda I|=0$$

$$\begin{bmatrix}1-\lambda & -1\\ -1 & 1-\lambda\end{bmatrix}=0$$

$$(1-\lambda)^2-1=0$$

$$[(1-\lambda)+1][(1-\lambda)-1]=0$$

$$\Rightarrow(2-\lambda)(-\lambda)=0$$

$$\lambda=2,\,0$$

Hence, eigenvalues of $A=2,\,0$

Now, using $|A-\lambda I|\widehat{X}=0$

Putting $\lambda=0$, we have

$$\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=0$$

$$x_1-x_2=0$$

$$-x_1+x_2=0$$

$$x_1=x_2$$

Putting $\lambda=2$, we have

$$\begin{bmatrix}-1 & -1\\ -1 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=0$$

$$-x_1-x_2=0$$

$$x_1=-x_2$$

Hence, from the given options, eigenvectors for the corresponding eigenvalues can be $\begin{bmatrix}1\\ 1\end{bmatrix}$ and $\begin{bmatrix}1\\ -1\end{bmatrix}$.

33. (a) We have

$$A=\begin{bmatrix}0 & -1 & 1\\ -1 & 0 & 0\\ 1 & 1 & 1\end{bmatrix}$$

Now,

$$|A-\lambda I|=\begin{vmatrix}0-\lambda & -1 & 1\\ -1 & 0-\lambda & 0\\ 1 & 1 & 1-\lambda\end{vmatrix}=0$$

$$\Rightarrow(0-\lambda)[(0-\lambda)(1-\lambda)-0]+1[(\lambda-1)-1]$$
$$+1[0-(0-\lambda)]=0$$

$$\Rightarrow-\lambda\left[-\lambda+\lambda^2\right]+1[\lambda-2]+1[\lambda]=0$$

$$\Rightarrow-\lambda^3+\lambda^2+\lambda-2+\lambda=0$$

$$\Rightarrow\lambda^3-\lambda^2-2\lambda+2=0$$

$$\Rightarrow\lambda^2(\lambda-1)-2(\lambda-1)=0$$

$$\Rightarrow(\lambda^2-2)(\lambda-1)=0$$

$$\Rightarrow\left(\lambda+\sqrt{2}\right)\left(\lambda-\sqrt{2}\right)(\lambda-1)=0$$

Hence, eigenvalues are $\lambda=\sqrt{2},-\sqrt{2},1$.

34. (b) We have

$$A=\begin{bmatrix}4 & 0 & 0\\ 1 & 4 & 0\\ 0 & 0 & 5\end{bmatrix}$$

Now,

$$(A - \lambda I) = \begin{vmatrix} 4-\lambda & 0 & 0 \\ 1 & 4-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (4-\lambda)[(4-\lambda)(5-\lambda)-0]-1[0-0]+0[0-0]=0$$

$$\Rightarrow (4-\lambda)(4-\lambda)(5-\lambda)=0$$

$\therefore$ Eigenvalues of the matrix are $\lambda = 4, 4, 5$.

35. (d) We have

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 6 & 3 \end{bmatrix}$$

The characteristic equation is given by

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 2-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)[(2-\lambda)(3-\lambda)-0]-0[(3-\lambda)(2-\lambda)-0]$$
$$+0[2-0]=0 \Rightarrow (1-\lambda)(2-\lambda)(3-\lambda)=0$$

$$\lambda = 1,\ 2,\ 3$$

Using $|A - \lambda I|\widehat{X} = 0$

and putting $\lambda = 1$, we get

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$y = 0$$
$$y - 2z = 0$$
$$2z = y$$
$$2z = 0$$

Hence, $x = k,\ y = 0,\ z = 0.$

Therefore, the eigenvector can be of the form $\begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}$.

Now putting $\lambda = 2$, we get

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x + y = 0 \qquad \Rightarrow x = y$$
$$2z = 0 \qquad \Rightarrow z = 0$$
$$z = 0$$

Hence, $x = k,\ y = k,\ z = 0.$

Therefore, the eigenvector can be of the form $\begin{bmatrix} k \\ k \\ 0 \end{bmatrix}$.

Now putting $\lambda = 3$, we get

$$\begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x + y = 0 \Rightarrow x = y/2$$
$$-y + 2z = 0 \Rightarrow y = 2z$$

Hence, $x = 2k,\ y = k,\ z = 2k.$

Therefore, the eigenvector can be of the form $\begin{bmatrix} 2k \\ k \\ 2k \end{bmatrix}$.

Hence, the eigenvector which is not of the matrix A is $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

36. (b) We have

$$A = \begin{bmatrix} 2 & 5 & 0 \\ 0 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

The characteristic equation is given by

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 5 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 1 & 1-\lambda \end{vmatrix}$$

$$\Rightarrow (2-\lambda)(3-\lambda)(1-\lambda) = 0$$

$$\lambda = 1, 2, 3$$

Now, using $|A - \lambda I|\widehat{X} = 0$

Putting $\lambda = 1$, we get

$$\begin{bmatrix} 1 & 5 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x + 5y = 0$$
$$2y = 0$$
$$y = 0$$

Thus, the eigenvector can be of the form $\begin{bmatrix} 0 \\ 0 \\ k \end{bmatrix}$.

Putting $\lambda = 2$, we get

$$\begin{bmatrix} 0 & 5 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$5y = 0$$
$$y = 0$$
$$y - z = 0 \Rightarrow y = z$$

Thus, the eigenvector can be of the form $\begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}$.