SIGNAL ENERGY AND POWER

v(t) and i(t) are continuous signal across R.

Instantaneous Power

$$p(t) = v(t)\,.\,i(t) = \left(\frac{1}{R}\right) v^2(t)$$

$$\text{The total energy} = \int_{t_1}^{t_2} p(t)\,dt = \int_{t_1}^{t_2} (1/R)\, v^2(t)\,dt$$

$$\text{Average power} = \frac{1}{t_1 - t_2} \int_{t_1}^{t_2} p(t)\,dt$$

For continuous signal x(t)

$$\text{Total energy} = \int_{t_1}^{t_2} |x(t)|^2\,dt$$

For discrete signal x[n]

$$\text{Total energy} = \sum_{n=n_1}^{n_2} |X[n]^2$$

Total energy of signal for $-\infty < t < +\infty$ or $-\infty < n < +\infty$

Continuous Time

$$E_\infty = \lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2\,dt = \int_{-\infty}^{\infty} |x(t)|^2\,dt$$

$$\text{Discrete Time } E_\infty = \lim_{T \to \infty} \sum_{n=-N}^{+N} |\,x[n]^2 = \sum_{n=-\infty}^{+N} |\,x[n]\,|^2$$

$$\text{Time average Power } P_\infty = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2\,dt$$

$$P_\infty = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n-\infty}^{N} |x(t)|^2$$

BASIC SYSTEM PROPERTIES

System without memory/static

A system is said to be memory less if its output for each value of the independent variable at a given time is dependent only on the input at the same time.

Example : $y[n] = \{2x[n] - x^2[n]\}^2$

For Resistor,

$x(t)$ = input current

$y(t)$ = output voltage

then, $y(t) = R\,x(t)$

For identity system

$y(t) = x(t)$; $y[n] = x[n]$

System with Memory/ dynamic : Presence of mechanism which retain information at time other than current i.e. capacitor and inductor.

Example : accumulator or summer

$$y[n] = \sum_{k=-\infty}^{n} n(k) = \sum_{k=-\infty}^{n-1} x(k) + x[n]$$

(2) Delay circuit $y[n] = x[n - 1]$

$= y[n - 1] + x[n]$

Invariability and Inverse System

A systems is said to be invertible if distinct input lead to distinct output. If a system is invertible, then a inverse system exist that when cascaded with the original system, yields an output w[n] equal to the input x[n] to the first system.

$$H_1 \rightarrow y[n] = \sum_{k=-\infty}^{n} x(k)$$

$$H_2 = w[n] = y[n] - y[n - 1]$$

Noninvertible System are

$y[n] = 0$ and $y(t) = x^2(t)$

1. The system that produce zero output sequence for input.
2. In which we cannot determine the sign of the input form knowledge of output.

Use : Encoder must be invertible in communication system.

Causality

A system is casual if the output at any time depends on value of the input at the present time and in the past. Such system is referred as non- anticipative as the system does not anticipate future values of the input.

The RC series circuit is casual as the capacitor voltage responds only to the present and past values of source voltage.

Example :

1. The accumulator and capacitor
2. All memory less system are casual, since the output responds only to the current value of input.

   Not casual if

   $y[n] = x[n] - x[n + 1]$

   $y(t) = x(t + 1)$

   Where, $x[n + 1]$ represent future input

Stability

A stable system is one in which small inputs lead to responds that do not diverge.

For the stable system, if the input to a stable system is bounded, then the output must also be bounded.

The stability of physical system generally results from the presence of mechanism that dissipate energy.

Examples :

1. $S_1 : y(t) = t\, x(t)$

   if $x(t) = u(t)$

   then $y(t) = t$ => unbounded => unstable
2. $S_2 : y(t) = e^{x(t)}$ if $x(t)$ is bounded i.e $|x(t)| < B$

   => bounded or => stable

Time Invariance

A system is time invariant if the behaviour and characteristics of the system are fixed over time.

For example, the RC series circuit is time invariant if the resistance and capacitance value are constant.

A system is time invariant if a time shift in the input result in the identical shift in the output signal.

If $x(t)$ input then $y(t)$ output

If $x(t - t_0)$ - input then $y(t - t_0)$ output

Linearity

A linear system in continuous time or discrete time, is a system that possesses the important property of super position. i.e.

1. The response to $x_1(t) + x_2(t)$ is $y_1(t) + y_2(t)$
2. The response to $ax_1(t)$ is a $y_1(t)$ where a is complex constant.

$$\text{Example : } \left.\begin{aligned} y(t) &= x^2(t) \\ y(t) &= \sin[x(t)] \end{aligned}\right\} \text{Not linear}$$

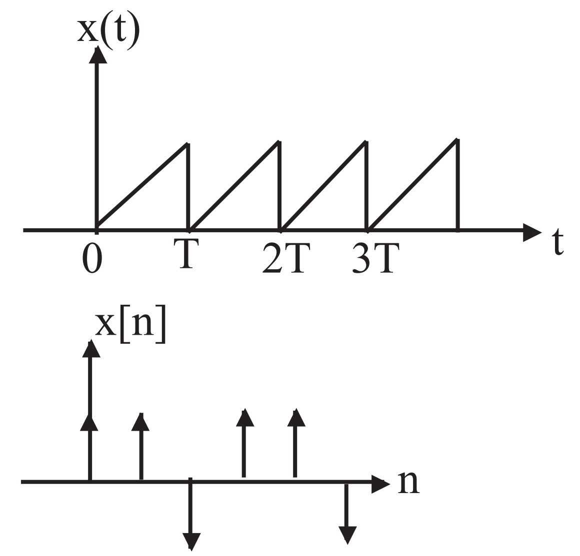
For linear system, an input which is zero for all time results in an output which is zero all time.

PERIODIC AND APERIODIC SIGNALS

A periodic signal has the property that it is unchanged by a time shift of T.

$$x(t) = x(t + T) \text{ and}$$
$$x[n] = x[n + N]$$

We can say that $x(t)$ is periodic with time period T.



Time period $N_0 = 3$

If this property is not satisfy then the signal is called aperiodic signal or non Periodic signal.

IMPULSE RESPONSE

The property of timr invariance tells us that the response the time -invariant system to the time shifted unit impulse are simply time shifted version of one another.

For any discrete LTI system output is given by

$$y[n] = x[n] * h[n]$$

Where, $x[n] = \sum_{k=-\infty}^{\infty} n(k)\, \delta[n-k]$

and $h[n] \rightarrow$ response of $\delta[n]$

$h_k[n] \rightarrow$ response of linear system to the shifted unit impulse $\delta[n-k]$

$$\therefore\ y[n] = \sum_{k=-\infty}^{\infty} x(k)\, h_k[n]$$

$$= \sum_{k=-\infty}^{\infty} x(k)\, h[n-k]$$

Note : If $x[n] \rightarrow$ has n element

$h[n] \rightarrow$ m element

Then $y[n] \rightarrow [m + n - 1]$ element

TRANSFER FUNCTION AND FREQUENCY RESPONSE OF FIRST AND SECOND ORDER SYSTEMS

Transfer Function of LTI Continuous Time System in Freqency Domain

The ratio of fourier transform of output and the Fourier transform of input is called transfer function of LTI continuous time system in frequency domain .

Let

$x(t)$ = Input to the continuous time system

$y(t)$ = Output of the continuous time system

$X(j\omega)$ = Fourier transform of x(t)
$Y(j\omega)$ = Fourier transform y(t)

Now, Transfer function $= \frac{Y(j\omega)}{X(j\omega)}$

Frequency Response of LTI Continuous Time System

The output y(t) of an LTI continuous time system is given by convolution of h(t) and x(t)

$y(t) = x(t) * h(t)$

$$= h(t) * x(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t - \tau)\, d\tau \qquad (1)$$

Consider a special class of input (sinusoidal input),

$Ae^{j\omega t} = A(\cos \omega t + j \sin \omega t)$

$x(t) = Ae^{j\omega t} \qquad (2)$

Where A = Amplitude,
ω = Angular frequency in rad/sec

$\therefore x(t - \tau) = Ae^{j\omega(t - \tau)} \qquad (3)$

On substituting for x(t - τ) from equation (1) in equation (2) we get,

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)\, Ae^{j\omega(t - \tau)}\, d\tau$$

The function H(jω) is complex quantity and so it can be expressed as magnitude frequency and phase function.

$H(j\omega) = |H(j\omega)| \angle H(j\omega)$

Where, $|H(j\omega)|$ = Magnitude function
$\angle H(j\omega)$ = Phase function

The sketch of magnitude function and phase function with respect to ω will be given the frequency response graphically.

$H(j\omega) = H_r(j\omega) + jH_i(j\omega)$

Where, $H_r(j\omega)$ = Real part of H(jω)
H(jω) Imaginary part of H(jω)

The magnitude function is defined as,

$|H(j\omega)|^2 = H(j\omega)\, H^*(j\omega)$

$= [H_r(j\omega) + jH_i(j\omega)][H_r(j\omega) - jH_i(j\omega)]$

Where, $H^*(j\omega)$ is complex conjugate of $H(j\omega)$

$\therefore\ |H(j\omega)|^2 = H_r^2(j\omega) + H_i^2(j\omega)$

$\Rightarrow |H(j\omega)| = \sqrt{H_r^2(j\omega) + H_i^2(j\omega)}$

The phase function is defined as

$$\angle H(j\omega) = Arg[H(j\omega)] = \tan^{-1}\left[\frac{H_i(j\omega)}{H_r(j\omega)}\right]$$

From equation $\left(H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}\right)$

CONVOLUTION

Convolution is a mathematical operation and is useful for describing the input output relationship in a linear time variant system. It is an important analytical tool for the communication engineers.

The convolution f(t) of two time functions $f_1(t)$ and $f_2(t)$ is defined by the following integral.

$$f(t) = \int_{\tau=-\infty}^{\infty} f_1(\tau)\, f_2(t - \tau)\, d\tau = f_1(t) * f_2(t)$$

$FT[f_1(t) * f_2(t)] = F_1(\omega) . F_2(\omega)$

Application

$FT[f_1(t) . f_2(t)] = (1/2\pi) [F_1(\omega) * F_2(\omega)]$

Physical Meaning of convolution

CONVOLUTION ON LAPLACE TRANSFORM BASIS

$$f_1(t) * f_2(t) = \int_0^t f_1(t - \tau) . f_2(\tau)\, d\tau$$

$$f_1(t) * f_2(t) = \int_0^t f_2(t - \tau) . f_1(\tau)\, d\tau$$

$f_1(t) * f_2(t) = f_2(t) * f_1(t)$

The word convolve means to roll together. The two function $f_1(t)$ and $f_2(t)$ are multiplied together in such a manner that one is continuously moving with time τ relative to the other.

DISCRETE CONVOLUTION

Convolution is important in digital signal processing because convolving two sequence in the time domain is equivalent to multiplying the sequence in the frequency domain. Convolution finds its application in processing signals especially analyzing the output of a system. Consider the signals $x_1[n]$ and $x_2[n]$. The convolution of these two signals is given by

$$x_3[n] = x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n - k)$$

ANALYSIS OF LTI DISCRETE TIME SYSTEM USING DISCRETE TIME FOURIER TRANSFORM (DTFT)

Transform Function of LTI Discrete Time System in Frequency Domain (DTFT)

The ratio of Fourier transform and output and the Fourier transform of input is called transfer function of LTI discrete time system is frequency domain.

Let, $x(n)$ = Input to the discrete time system

$y(n)$ = Output of the discrete time system

$\therefore$ $X(e^{j\omega})$ = Fourier transform of $x(n)$

$Y(e^{j\omega})$ = Fourier transform of $y(n)$

$$\text{Now, Transfer function} = \frac{Y(e^{j\omega})}{X(e^{j\omega})} \qquad (1)$$

The transfer function of an LTI discrete time system in frequency domain can be obtained from the difference equation governing the input-output relation of the LTI discrete time system given below

$$y(n) = -\sum_{m-1}^{N} a_m\, y(n - m) + \sum_{m=0}^{M} b_m\, x(n - m)$$

On taking Fourier transform of above equation and rearranging the resultant equation as a ratio of $Y(e^{j\omega})$ and $X(e^{j\omega})$, the transfer function of LTI discrete time system in frequency domain is obtained.

Properties of continuous time and Discrete time LTI systems

We know that continuous time and discrete time systems are characterized by their unit impulse responses. Response of continuous time LTI system is determined by using convolution integral.

While the response of discrete time LTI system is determined by using convolution sum.

Now, we will discuss various properties of continuous time and discrete time LTI systems.

Convolution integral for continuous time LTI is given by

$$y(t) = x(t) * h(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\, h(t - \tau)\, d\tau$$

and convolution sum for discrete time LTI system is given by

$$y[n] = x[n] * h[n] = \sum_{K=-\infty}^{\infty} x(k)\, h(n - k)$$

Both continuous time and discrete time LTI systems are completely characterized by their impulse response. These systems are also completely characterized by their transfer function. For continuous time LTI system, the transfer function is the ratio of Laplace transform of output to Laplace transform of input to the system when system is initially relaxed. It is also defined as Laplace transform of impulse of a continuous time LTI system.

For discrete time LTI system, transfer function is the ratio of z-transform of output to z-transform of input when initially system is relaxed. It is also defined, as z-transform of unit impulse response of discrete time LTI system.

Causality of LTI Systems

The output of a causal system depends only on the present and past values of the input to the system.

A continuous tie LTI system is called causal system if its impulse response h(t) is zero for $t < 0$.

For a causal continuous time LTI system convolution integral is given by

$$y(t) = x(t) * h(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\, h(t - \tau)\, d\tau$$

$$= h(t) * x(t) = \int_{\tau=0}^{\infty} h(\tau)\, x(t - \tau)\, d\tau$$

A discrete time LTI system is called causal system if its impulse response h[n] is zero for $n < 0$.

For a causal discrete time LTI system, convolution sum is given by

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{n} x(k)\, h(n - k)$$

$$= h[n] * x[n] = \sum_{k=0}^{\infty} h(k)\, x(n - k)\, d\tau$$

Causality of an LTI system is equivalent to its impulse response being a causal signal. Accumulator with impulse response.

$$h[n] = u[n] = \begin{cases} 0 & n < 0 \\ 1 & n < 0 \end{cases}$$

is a causal discrete time LTI system.

A pure time shift with unit impulse response $h(t) = \delta(t - t_0)$ is a causal continuous time LTI system for $t \geq 0$. In this case time shift is called a delay.

A pure time shift is non causal continuous time LTI system for $t < 0$. In this case time shift is called an advance.

Stability for LTI Systems

A stable system produces bounded output from every bounded input.

Condition of stability for continuous time LTI system :

Consider an input x(t) that is bounded in magnitude.

$| x(t) | < M$ for all values of t (1)

Now we apply this input to an continuous time LTI system with unit impulse response h(t).

Output of this LTI system is determined by convolution integral and is given by

$$y(t) = \int_{t=-\infty}^{\infty} h(\tau)\, x(t - \tau)\, d\tau \qquad (2)$$

Magnitude of output y(t) is given by

$$|y(t)| = \left| \int_{t=-\infty}^{\infty} h(\tau)\, x(t - \tau)\, d\tau \right|$$

$$= \int_{t=-\infty}^{\infty} | h(\tau) |\, | x(t - \tau)|\, d\tau \qquad (3)$$

[Substituting the value $|s\,(t - \tau)| < M$ for all values of τ and t].

$$\text{or } | y(t) | \leq \int_{t=-\infty}^{\infty} |h(\tau)|\, .\, M.\, d\tau$$

$$\leq M \int_{t=-\infty}^{\infty} |h(\tau)|\, d\tau \text{ for all values of } t \qquad (4)$$

From equation we can conclude that if the impulse response h(t) is absolutely integerable than output of a continuous time LTI system is bounded in magnitude, and hence, the system is bounded input bounded-output (BIBO) stable.

A sufficient and necessary condition to stability of a continuous time LTI system is given by

$$S = \int_{\tau=-\infty}^{\infty} |h(\tau)|\, d\tau < \infty$$

Condition to stability of a discrete time LTI system

Consider an input x[n] that is bounded in magnitude.

$|x[n]| < M$ for all values of n (1)

Now we apply this input to a discrete time LTI system with unit impulse response h(n). Output of discrete time LTI system is determined by convolution sum and is given by

$$y[n] = \sum_{K=-\infty}^{\infty} h(k)\, s(n - k) \qquad (2)$$

Magnitude of y(n) is given by

$$|y[n]| = \left| \sum_{k=\infty}^{\infty} h(k)\, x(n - k) \right|$$

$$= \sum_{k=-\infty}^{\infty} |h(k)|\, |x(n - k)| \qquad (3)$$

[Substituting the value of $|x(n - k)| < M$ for all value of k and n].

$$\text{or } |\, y[n]\, | \leq \sum_{K=-\infty}^{\infty} |\, h(k)\, | M$$

$$\leq M \sum_{K=-\infty}^{\infty} |\, h(k)\, | \qquad (4)$$

for all values of k.

From equation we can conclude that if the impulse response h(n) is absolutely summable then output of a discrete time LTI system is bounded input.

IMPULSE AND FREQUENCY RESPONSE (LTI System)

IMPULSE RESPONSE

The unit sample response or simply the impulse response h[n] of a linear time invarient system is the system's response to a unit impulse input signal $\{\delta[n]\}$ located at $n = 0$, when the initial conditions of the system are zero.

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$