CHAPTER 2

CALCULUS

LIMITS

Limits are used to define continuity, derivatives and integrals.

Consider a function $f(x)$. Now, if x approaches a value c, and if for any number $\alpha > 0$, we find a number $\beta > 0$ such that $|f(x) - l| < \alpha$ whenever $0 < |x - c| < \beta$, then l is called the limit of function $f(x)$.

Limits are denoted as

$$\lim_{x \to c} f(x) = l$$

For example, if we have a function $f(x) = \dfrac{x^2 - 4}{x - 2}$, its limit does not exist for $x = 2$ since $f(1) = \infty$. However, as x moves closer or approaches the value 2, $f(x)$ approaches the limit 4.

Left-Hand and Right-Hand Limits

If the values of a function $f(x)$ at $x = c$ can be made as close as desired to the number l_1 at points closed to and on the left of c, then l_1 is called left-hand limit.

It is denoted by

$$\lim_{x \to c^-} f(x) = l_1$$

If the values of a function $f(x)$ at $x = c$ can be made as close as desired to the number l_2 at points on the right of and close to c, then l_2 is called right-hand limit.

It is denoted by

$$\lim_{x \to c^+} f(x) = l_2$$

For example, let us calculate the left-hand and right-hand limits of the following function:

$$f(x) = \begin{cases} \dfrac{|x-1|}{x-1}, & x \neq 1 \\ 0 \quad , & x = 1 \end{cases} \quad \text{at } x = 1$$

L.H.L. of $f(x)$ at $x = 1$

$$= \lim_{x \to 1^-} f(x) = \lim_{h \to 0} f(1-h)$$

$$= \lim_{h \to 0} \frac{|1-h-1|}{1-h-1} = \lim_{h \to 0} \frac{|-h|}{-h}$$

$$= \lim_{h \to 0} \frac{h}{-h} = \lim_{h \to 0} (-1) = -1$$

R.H.L. of $f(x)$ at $x = 1$

$$= \lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h)$$

$$= \lim_{h\to 0} \frac{|1+h-1|}{1+h-1} = \lim_{h\to 0} \frac{|h|}{h}$$

$$= \lim_{h\to 0} 1 = 1$$

PROPERTIES OF LIMITS

Suppose we have $\lim_{x\to c} f(x) = a, \lim_{x\to c} g(x) = b$ and if a and b exist, some of the important properties of limits are:

1. $\lim_{x\to c}(f(x) \pm g(x)) = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x) = a \pm b$
2. $\lim_{x\to c} (f(x)\cdot g(x)) = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x) = a \cdot b$
3. $\lim_{x\to c}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{a}{b}$, where $b \neq 0$
4. $\lim_{x\to c} kf(x) = k \lim_{x\to c} f(x)$, where k is a constant
5. $\lim_{x\to c} |f(x)| = |\lim_{x\to c} f(x)| = |a|$
6. $\lim_{x\to c} (f(x))g(x) = a^b$
7. If $\lim_{x\to c} f(x) = \pm\infty$, then $\lim_{x\to c} \frac{1}{f(x)} = 0$

Some of the useful results of limits are given as follows:

1. $\lim_{x\to 0} \frac{\sin x}{x} = 1$
2. $\lim_{x\to 0} \frac{\tan x}{x} = 1$
3. $\lim_{x\to 0} \cos x = 1$
4. $\lim_{x\to 0}(1+x)^{1/x} = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$
5. $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$
6. $\lim_{x\to 0} \frac{(a^x-1)}{x} = \log a$, if$(a > 0)$
7. $\lim_{x\to 0} \frac{x^n - a^n}{x-a} = na^{n-1}$
8. $\lim_{x\to\infty} \frac{\log x}{x^m} = 0$, if $(m > 0)$

L'Hospital's Rule

If $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ or $\pm\infty$, $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ exists and $g'(x) \neq 0$ for all x, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

CONTINUITY AND DISCONTINUITY

A function $f(x)$ at any point $x = c$ is continuous if

$$\lim_{x\to c} f(x) = f(c)$$

and $$\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = f(c)$$

A function $f(x)$ is continuous for an open interval (a, b) if it is continuous at every point on the interval (a, b).

A function $f(x)$ is continuous on a closed interval $[a, b]$ if

1. f is continuous for (a, b)
2. $\lim_{x\to a^+} f(x) = f(a)$
3. $\lim_{x\to b^-} f(x) = f(b)$

If the conditions for continuity are not satisfied for a function $f(x)$ for a point or an interval, then the function is said to be discontinuous.

DIFFERENTIABILITY

Consider a real-valued function $f(x)$ defined on an open interval (a, b). The function is said to be differentiable for $x = c$, if

$$\lim_{h\to 0} \frac{f(c+h) - f(c)}{h} \text{ exists for every } c \in (a, b)$$

A function is always continuous at a point if the function is differentiable at the same point. However, the converse is not always true.

MEAN VALUE THEOREMS

Rolle's Theorem

Consider a real-valued function defined in the closed interval $[a, b]$, such that

1. It is continuous on the closed interval $[a, b]$.
2. It is differentiable on the open interval (a, b).
3. $f(a) = f(b)$.

Then, according to Rolle's theorem, there exists a real number $c \in (a,b)$ such that $f'(c) = 0$.

Lagrange's Mean Value Theorem

Consider a function $f(x)$ defined in the closed interval $[a, b]$, such that

1. It is continuous on the closed interval $[a, b]$.
2. It is differentiable on the closed interval (a, b).

Then, according to Lagrange's mean value theorem, there exists a real number $c \in (a,b)$, such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Cauchy's Mean Value Theorem

Consider two functions $f(x)$ and $g(x)$, such that

1. $f(x)$ and $g(x)$ both are continuous in $[a, b]$.
2. $f'(x)$ and $g'(x)$ both exist in (a, b).

Then there exists a point $c \in (a,b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Taylor's Theorem

If $f(x)$ is a continuous function such that $f'(x)$, $f''(x), \ldots, f^{n-1}(x)$ are all continuous in $[a, a + h]$ and $f^n(x)$ exists in $(a, a + h)$ where $h = b - a$, then according to Taylor's theorem,

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} f^n(a)$$

Maclaurin's Theorem

If the Taylor's series obtained in Section 2.5.4 is centered at 0, then the series we obtain is called the Maclaurin's series. According to Maclaurin's theorem,

$$f(h) = f(0) + hf'(0) + \frac{h^2}{2!} f''(0) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(0) + \frac{h^n}{n!} f^n(0)$$

FUNDAMENTAL THEOREM OF CALCULUS

There are two parts of the fundamental theorem of calculus that are used widely. It links the concept of the derivative of the function with the concept of the integral.

According to the first part of the fundamental theorem of calculus or the first fundamental theorem of calculus, if f is continuous on the closed interval $[a, b]$ and F is the indefinite integral of f on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

According to the second part of the fundamental theorem of calculus or the second fundamental theorem of calculus, if f is a continuous real-valued function defined on a closed interval $[a, b]$ and F is a function defined for all x in $[a, b]$, by

$$F(x) = \int_a^x f(t)dt$$

then F is continuous on $[a, b]$, differentiable on the open interval (a, b) and $F'(x) = f(x)$ for all x in (a, b).

DIFFERENTIATION

Some of the important properties of differentiation are:

1. $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$
2. $\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx} f(x) - \frac{d}{dx} g(x)$
3. $\frac{d}{dx}(f(x) \cdot g(x)) = f(x) \cdot \frac{d}{dx}(g(x)) + \frac{d}{dx}(f(x)) \cdot g(x)$
4. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{d}{dx} f(x) \cdot g(x) - f(x) \frac{d}{dx} g(x)}{(g(x))^2}$

Some of the derivatives of commonly used functions are given as follows:

1. $\frac{d}{dx} x^n = nx^{n-1}$
2. $\frac{d}{dx} \ln x = \frac{1}{x}$
3. $\frac{d}{dx} \log_a x = \log_a e \cdot \left(\frac{1}{x}\right)$

4. $\frac{d}{dx}e^x = e^x$

5. $\frac{d}{dx}a^x = a^x \log_e a$

6. $\frac{d}{dx}\sin x = \cos x$

7. $\frac{d}{dx}\cos x = -\sin x$

8. $\frac{d}{dx}\tan x = \sec^2 x$

9. $\frac{d}{dx}\sec x = \sec x \tan x$

10. $\frac{d}{dx}\text{cosec}\, x = -\text{cosec}\, x \cot x$

11. $\frac{d}{dx}\cot x = -\text{cosec}^2\, x$

12. $\frac{d}{dx}\sin \text{h}\, x = \cos \text{h}\, x$

13. $\frac{d}{dx}\cos \text{h}\, x = \sin \text{h}\, x$

14. $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$

15. $\frac{d}{dx}\cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}$

16. $\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$

17. $\frac{d}{dx}\text{cosec}^{-1} x = -\frac{1}{x\sqrt{x^2-1}}$

18. $\frac{d}{dx}\sec^{-1} x = -\frac{1}{x\sqrt{x^2-1}}$

19. $\frac{d}{dx}\cot^{-1} x = \frac{1}{1+x^2}$

APPLICATIONS OF DERIVATIVES

Increasing and Decreasing Functions

Any function $f(x)$ is said to be increasing on an interval (a, b) if

$$x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2) \text{ for all values of } x_1, x_2 \in (a, b)$$

A function $f(x)$ is said to be strictly increasing on an interval (a, b) if

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \text{ for all values of } x_1, x_2 \in (a, b)$$

A function $f(x)$ is said to be decreasing on an interval (a, b) if

$$x_1 \leq x_2 \Rightarrow f(x_1) \geq f(x_2) \text{ for all values of } x_1, x_2 \in (a, b)$$

A function $f(x)$ is said to be strictly decreasing on an interval (a, b) if

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2) \text{ for all values of } x_1, x_2 \in (a, b)$$

A monotonic function is any function $f(x)$ which is either increasing or decreasing on an interval (a, b).

Some important conditions for increasing and decreasing functions are:

1. Consider $f(x)$ to be continuous on $[a, b]$ and differentiable on (a, b). Now,
   (a) If $f(x)$ is strictly increasing on (a, b), then $f'(x) > 0$ for all $x \in (a, b)$.
   (b) If $f(x)$ is strictly decreasing on (a, b), then $f'(x) < 0$ for all $x \in (a, b)$.
2. Consider $f(x)$ to be a differentiable real function defined on an interval (a, b). Now,
   (a) If $f'(x) > 0$ for all $x \in (a, b)$, then $f(x)$ is increasing on (a, b).
   (b) If $f'(x) < 0$ for all $x \in (a, b)$, then $f(x)$ is decreasing on (a, b).

For example, let us find the intervals for which $f(x) = x^4 - 2x^2$ is increasing or decreasing.

$$f(x) = x^4 - 2x^2$$

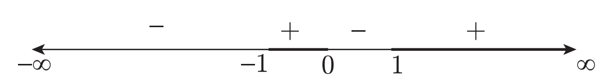
$$f'(x) = 4x^3 - 4x = 4x(x^2 - 1)$$

For $f(x)$ to be increasing, $f'(x) > 0$

$\therefore$ $4x\,(x^2 - 1) > 0$

$\Rightarrow x\,(x^2 - 1) > 0$

$\Rightarrow x(x + 1)(x - 1) > 0$



$\Rightarrow -1 < x < 0$ or $x > 1$

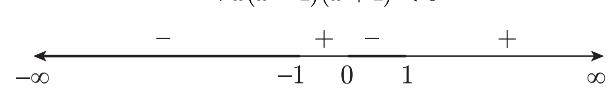
$\therefore x \in (-1, 0) \cup (1, \infty)$

For $f(x)$ to be decreasing, $f'(x) < 0$

$\Rightarrow 4x(x^2 - 1) < 0$

$\Rightarrow x(x^2 - 1) < 0$

$\Rightarrow x(x - 1)(x + 1) < 0$



$\Rightarrow x < -1$ or $0 < x < 1$

$\therefore x \in (-\infty, -1) \cup (0, 1)$

Maxima and Minima

Suppose $f(x)$ is a real-valued function defined at an internal (a, b). Then $f(x)$ is said to have maximum value, if there exists a point y in (a, b) such that

$$f(x) = f(y) \text{ for all } x \in (a, b)$$

Suppose $f(x)$ is a real-valued function defined at the interval (a, b). Then $f(x)$ is said to have minimum value, if there exists a point y in (a, b) such that

$$f(x) \geq f(y) \text{ for all } x \in (a, b)$$

Local maxima and local minima of any function can be calculated as:

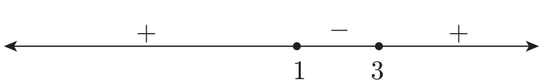
Consider that $f(x)$ be defined in (a, b) and $y \in (a, b)$. Now,

1. If $f'(y) = 0$ and $f'(x)$ changes sign from positive to negative as 'x' increases through 'y', then $x = y$ is a point of local maximum value of $f(x)$.
2. If $f'(y) = 0$ and $f'(x)$ changes sign from negative to positive as 'x' increases through 'y', then $x = y$ is a point of local minimum value of $f(x)$.

For example,

1. $f(x) = x^3 - 6x^2 + 12x$
   Find all points of local maxima and local minima.
   $$y = f(x) = x^3 - 6x^2 + 12x$$
   $$\frac{dy}{dx} = 3x^2 - 12x + 12 = 3(x-2)^2$$
   For a local maximum or local minimum, we have
   $$\frac{dy}{dx} = 0 \Rightarrow 3(x-2)^2 = 0 \Rightarrow x = 2$$
   We observe that dy/dx does not change sign as increased through $x = 2$. Hence, $x = 2$ is neither a point of local maximum nor local minimum.

2. Find all points of local maxima and minima of the function,
   $$f(x) = x^3 - 6x^2 + 9x$$
   Now, $f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3)$
   For a local maximum or local minimum, we have
   $$f'(x) = 0 \Rightarrow 3(x^2 - 4x + 3) = 0 \Rightarrow x = 1, 3$$
   The change in signs of $f'(x)$ are shown in the following figure.



   $f'(x)$ changes sign from positive to negative as 'x' increases through 1.

   Therefore, $x = 1$ is a point of local maximum.

   Also, $f'(x)$ changes sign from negative to positive as 'x' increases through 3.

   Therefore, $x = 3$ is a point of local minimum.

Some important properties of maximum and minima are given as follows:

1. If $f(x)$ is continuous in its domain, then at least one maxima and minima lie between two equal values of x.
2. Maxima and minima occur alternately, i.e. no two maxima or minima can occur together.

For example, let us find all points of maxima and minima for the following points:

$$f(x) = 2x^3 - 21x^2 + 36x$$

Now, $f'(x) = 6x^2 - 42x + 36$

For local maxima or minima,

$$f'(x) = 0$$

$\therefore 6x^2 - 42x + 36 = 0$

$\Rightarrow (x-1)(x-6) = 0$

$\Rightarrow x = 1, 6$

Now, to test for maxima and minima,

$$f''(x) = 12x - 42$$

At $x = 1$,

$f''(1) = -30 < 0$, hence $x = 1$ is a point of local maximum.

At $x = 6$,

$f''(6) = 72 - 42$

$= 30 > 0$, hence $x = 6$ is a point of local minimum.

Maximum and minimum values in a closed interval $[a, b]$ can be calculated using the following steps:

1. Calculate $f'(x)$.
2. Put $f'(x) = 0$ and find value(s) of x. Let $c_1, c_2, \ldots, c_n$ be values of x.
3. Take the maximum and minimum values out of the values $f(a), f(c_1), f(c_2), \ldots, f(c_n), f(b)$. The maximum and minimum values obtained are the absolute maximum and absolute minimum values of the function, respectively.

For example, let us find the points of maxima and minima for $f(x) = 2x^3 - 24x + 107$ in the interval $[0, 3]$.

$$f'(x) = 6x^2 - 24$$

Now, $f'(x) = 0$

$\Rightarrow 6x^2 - 24 = 0 \Rightarrow x^2 = \frac{24}{6} = 4$

$\therefore x = \pm 2$

But, $x = -2 \notin [0, 3]$

$\therefore\ x = 2$ is the only stationary point.

$f(0) = 107$

$f(2) = 2(2)^3 - 24(2) + 107$

$= 16 - 48 + 107 = 75$

$f(3) = 2(3)^3 - 24(3) + 107 = 89$

The maximum value of $f(x)$ is 107 at $x = 0$.

The minimum value of $f(x)$ is 75 at $x = 2$.

Therefore, the points of maxima and minima are 0 and 2, respectively.

PARTIAL DERIVATIVES

Partial differentiation is used to find the partial derivative of a function of more than one independent variable.

The partial derivatives of $f(x, y)$ with respect to x and y are defined by

$$\frac{\partial f}{\partial x} = \lim_{a \to 0} \frac{f(x + ay) - f(x, y)}{a}$$

$$\frac{\partial f}{\partial x} = \lim_{b \to 0} \frac{f(x, y + b) - f(x, y)}{b}$$

and the above limits exist.

$\partial f/\partial x$ is simply the ordinary derivative of f with respect to x keeping y constant, while $\partial f/\partial x$ is the ordinary derivative of f with respect to y keeping x constant.

Similarly, second-order partial derivatives can be calculated by

$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right), \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right), \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$ and is, respectively, denoted by $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial y^2}$.

A homogenous function is an expression in which every term is of the same degree. Thus, a homogeneous function of x and y of degree n can be represented as

$$a_0 x^n + a_1 x^{n-1} y + a_1 x^{n-2} y^2 + \cdots + a_n y^n$$

Euler's theorem on homogeneous function $f(x, y)$ of degree 'n' is given by

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

If $u = f(x, y)$ where $x = g_1(t)$ and $y = g_2(t)$, then

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t}$$

In the above equation, the term $\frac{\partial u}{\partial t}$ is called total differential coefficient of u with respect to t while $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are partial derivatives of u. Some of the important results from the above relation are given as follows:

1. If $u = f(x, y)$ and $y = f(x)$, then

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial x}$$

2. If $u = f(x, y)$ and $x = f_1(t_1, t_2)$ and $y = f_2(t_1, t_2)$, then

$$\frac{\partial u}{\partial t_1} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t_1} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t_1}$$

and

$$\frac{\partial u}{\partial t_2} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t_2} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t_2}$$

INTEGRATION

We have already discussed differentiation in the previous sections of this chapter. In this section, we discuss the other main operation of calculus, integration.

Given a function f of a real variable x and an interval $[a, b]$ of the real line, the definite integral $\int_a^b f(x)dx$ is defined as the area of the region in the xy-plane that is bounded by the graph of f, the x-axis and the vertical lines $x = a$ and $x = b$.

METHODS OF INTEGRATION

Integration, unlike differentiation, is not straightforward. Some of the integrals can be solved directly from the table, however, in most of the calculations we need to apply one or the other techniques of integration. In this section, we discuss those techniques in order to make the integration problems easier to solve.

Integration Using Table

Some of the common integration problems can be solved by directly referring the tables and computing the results. Table 1 shows the result of some of the common integrals we use.

Table 1 | Table of common integrals

Integration	Result
$\int \frac{1}{ax+b}dx$	$\frac{1}{a}\ln\lvert ax+b\rvert + C$ where C is a constant
$\int \frac{1}{(ax+b)^2}dx$	$-\frac{1}{a(ax+b)}+C$
$\int \frac{1}{(ax+b)^n}dx$	$-\frac{1}{a(n-1)(ax+b)^{n-1}}+C$
$\int \frac{1}{a^2+x^2}dx$	$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$
$\int \frac{f'(x)}{f(x)}dx$	$\ln\lvert f(x)\rvert + C$
$\int \sin^2 x dx$	$\frac{x}{2}-\frac{1}{2}\sin x\cos x + C$
$\int \sin^3 x dx$	$-\cos x + \frac{1}{3}\cos^3 x + C$
$\int \sin^n x dx$	$-\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} xdx + C$
$\int \cos^2 x dx$	$\frac{x}{2}+\frac{1}{2}\sin x\cos x + C$
$\int \cos^3 x dx$	$\sin x - \frac{1}{3}\sin^3 x + C$
$\int \cos^n x dx$	$\frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} xdx + C$
$\int \cos^n x dx$	$\frac{\tan^{-1} x}{n-1} - \int \tan^{n-2} xdx + C$
$\int \frac{dx}{x^2-a^2}$	$\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert + C$
$\int \frac{dx}{\sqrt{x^2 \pm a^2}}$	$\ln\left\lvert x+\sqrt{x^2\pm a^2}\right\rvert + C$
$\int x\sin nx dx$	$\frac{1}{n^2}(\sin nx - nx\cos nx) + C$
$\int x\cos nx dx$	$\frac{1}{n^2}(\cos nx + nx\sin nx) + C$

(Continued)

Table 1 | Continued

Integration	Result
$\int e^{ax}\sin bx dx$	$\frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}+C$
$\int e^{ax}\cos bx dx$	$\frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}+C$
$\int x^2\sin nx dx$	$\frac{1}{n^3}(-n^2x^2\cos n x + 2\cos nx + 2nx\sin nx) + C$
$\int x^2\cos nx dx$	$\frac{1}{n^3}(n^2x^2\sin n x - 2\sin nx + 2nx\cos nx) + C$

Integration Using Substitution

There are occasions when it is possible to perform integration using a substitution to solve a complex piece of integration. This has the effect of changing the variable, the integrand and even the limits of integration (for definite integrals).

To integrate a differential $f(x)dx$ which is not in the table, we first take a function $u = u(x)$ so that the given differential can be rewritten as a differential $g(u)du$ which does appear in the table. Then, if $\int g(u)du = G(u)+C$, we know that

$$\int f(x)dx = G(u(x)) + C.$$

Integration by Parts

Sometimes we can recognize the differential to be integrated as a product of a function which is easily differentiated and a differential which is easily integrated.

Integration by parts is a technique for performing integration (definite and indefinite) by expanding the differential of a product of function $d(uv)$ and expressing the original integral in terms of a known integral $\int vdu$.

Using the product rule for differentiation, we have

$$d(uv) = udv + vdu$$

Integrating both sides, we get

$$\int d(uv) = uv = \int udv + \int vdu$$

Rearranging the above equation, we get

$$\int udv = uv - \int vdu$$

Integration by Partial Fraction

If the integrand is in the form of an algebraic fraction and the integral cannot be evaluated by simple methods, the fraction needs to be expressed in *partial fractions* before integration takes place.

The point of the partial fractions expansion is that integration of a rational function can be reduced to the following formulae, once we have determined the roots of the polynomial in the denominator. The formula which come handy while working with partial fractions are given as follows:

$$\int \frac{1}{x-a}\,dx = \ln(x-a) + C$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{x}{a^2+x^2}\,dx = \frac{1}{2}\ln\left(a^2+x^2\right) + C$$

Integration Using Trigonometric Substitution

Trigonometric substitution is used to simplify certain integrals containing radical expressions. Depending on the function we need to integrate, we substitute one of the following expressions to simplify the integration:

(a) For $\sqrt{a^2-x^2}$, use $x = a\sin\theta$.

(b) For $\sqrt{a^2+x^2}$, use $x = a\tan\theta$.

(c) For $\sqrt{x^2-a^2}$, use $x = a\sec\theta$.

DEFINITE INTEGRALS

If a function $f(x)$ is defined in the interval $[a, b]$, then the definite integral of the function is given by

$$\int_a^b f(x)\cdot dx = [F(x)]_a^b = F(b) - F(a)$$

where $F(x)$ is an integral of $f(x)$, a is called the lower limit and b is the upper limit of the integral.

Geometrically, a definite integral represents the area bounded by curve $y = f(x)$, x-axis and the lines $x = a$ and $x = b$.

Some of the important properties of definite integrals are given as follows:

1. The value of definite integrals remains the same with change of variables of integration provided the limits of integration remain the same.

$$\int_a^b f(x)\cdot dx = \int_a^b f(y)\cdot dy$$

2. $\int_a^b f(x)\cdot dx = -\int_b^a f(x)\cdot dx$

3. $\int_a^b f(x)\cdot dx = \int_a^c f(x)\cdot dx + \int_c^b f(x)\cdot dx \quad \text{if } a < c < b$

4. $\int_0^{2a} f(x)\cdot dx = \int_0^a f(x)\cdot dx + \int_0^a f(2a-x)\cdot dx$

5. $\int_0^a f(x)\cdot dx = \int_0^a f(a-x)\cdot dx$

6. $\int_{-a}^a f(x)\cdot dx = 2\int_0^a f(x)\cdot dx$, if the function is even.

$\int_{-a}^a f(x)\cdot dx = 0$, if the function is odd.

7. $\int_0^{na} f(x)\cdot dx = n\int_0^a f(x)\cdot dx \quad \text{if } f(x) = f(x+a)$

IMPROPER INTEGRALS

An improper integral is a definite integral that has either or both limits infinite or an integrand that approaches infinity at one or more points in the range of integration. Improper integrals cannot be computed using a normal Riemann integral.

Such an integral is often written symbolically like a standard definite integral, perhaps with infinity as a limit of integration.

$$\lim_{b\to\infty}\int_a^b f(x)dx, \quad \lim_{a\to-\infty}\int_a^b f(x)dx$$

Figure 1 shows the graph of improper integral $\int_0^\infty \frac{dx}{1+x^2}$.

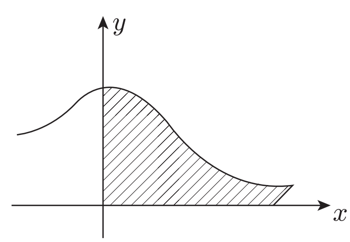


Figure 1 | An improper integral of the first kind.

The integral, $\int_1^{\infty} x^{-2}dx$ is an example of improper integral. This can be solved as follows:

$$\int_1^{y} x^{-2}dx = 1 - \frac{1}{y}$$

$$\Rightarrow \int_1^{\infty} x^{-2}dx = \lim_{y\to\infty}\int_1^{y} x^{-2}dx = \lim_{y\to\infty}\left(1-\frac{1}{y}\right) = 1$$

Improper integrals of the generalized form $\int_a^b f(x)dx$ with one of the limits being infinite and the other being non-zero may also be expressed as finite integrals over transformed functions. Let

$$t = \frac{1}{x}$$

Differentiating both sides, we get

$$dt = -\frac{dx}{x^2}$$

$$dx = -x^2 dt = -\frac{dt}{t^2}$$

Substituting the value of dx in the generalized form, we get

$$\int_a^b f(x)dx = -\int_{1/a}^{1/b} \frac{1}{t^2} f\left(\frac{1}{t}\right)dt = \int_{1/b}^{1/a} \frac{1}{t^2} f\left(\frac{1}{t}\right)dt$$

Now, considering that $f(x)$ diverges as $(x-a)^{\gamma}$ for $\gamma \in [0,1]$, let

$$x = t^{1/(1-\gamma)} + a$$

Differentiating both sides, we get

$$dx = \frac{1}{(1-\gamma)} t^{[1/(1-\gamma)-1]}dt = \frac{1}{(1-\gamma)} t^{[1-(1-\gamma)]/(1-\gamma)}dt$$

$$= \frac{1}{(1-\gamma)} t^{\gamma/(1-\gamma)}dt$$

$$t = (x-a)^{1-\gamma}$$

$$\Rightarrow \int_a^b f(x)dx = \frac{1}{1-\gamma} = \int_0^{(b-a)^{1-\gamma}} t^{\gamma/(1-\gamma)} f\left(t^{1/(1-\gamma)} + a\right)dt$$

Now, considering that $f(x)$ diverges as $(x+b)^{\gamma}$ for $\gamma \in [0,1]$, let

$$x = b - t^{1/(1-\gamma)}$$

Differentiating both sides, we get

$$dx = -\frac{1}{(\gamma-1)} t^{[\gamma/(1-\gamma)]}dt$$

$$t = (b-x)^{1-\gamma}$$

$$\Rightarrow \int_a^b f(x)dx = \frac{1}{1-\gamma} = \int_0^{(b-a)^{1-\gamma}} t^{\gamma/(1-\gamma)} f\left(b - t^{1/(1-\gamma)}\right)dt$$

DOUBLE INTEGRATION

Suppose we have a function $f(x, y)$ defined in a closed area Q of xy plane. Now, we can divide Q into n sub-regions ΔQ_k of area ΔA_k, $k = 1, 2, \ldots, n$. Let (a_k, b_k) be any arbitrary point of ΔQ_k. Hence, the sum can be given as

$$\sum_{k=1}^{n} f(a_k, b_k)\,\Delta A_k$$

Now, let us consider

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(a_k, b_k)\,\Delta A_k$$

where limit is taken such that 'n' increases indefinitely and the largest linear dimension of each ΔQ_k approaches zero.

If the limit exists, then the double integral of $f(x, y)$ over region Q is denoted by

$$\iint_Q f(x, y)dA$$

Some of the important properties of double integrals are:

1. When x_1, x_2 are functions of y and y_1, y_2 are constants, then $f(x, y)$ is integrated with respect to x keeping y constant within the limits x_1, x_2 and the resulting expression is integrated with respect to y between the limits y_1, y_2.

$$\iint_Q f(x,y)dxdy = \int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y)dxdy$$

2. When y_1, y_2 are functions of x and x_1, x_2 are constants, $f(x, y)$ is first integrated with respect to y, keeping x constant and between the limits y_1, y_2 and the resulting expression is integrated with respect to x within the limits x_1, x_2.

$$\iint_Q f(x,y)dxdy = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x,y)dydx$$

3. When x_1, x_2, y_1 and y_2 are constants, then

$$\iint_Q f(x,y)dxdy = \int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y)dxdy = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x,y)dydx$$

CHANGE OF ORDER OF INTEGRATION

As already discussed, if limits are constant

$$\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y)dxdy = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x,y)dydx$$

Hence, in a double integral, the order of integration does not change the final result provided the limits are changed accordingly.

However, if the limits are variable, the change of order of integration changes the limits of integration.

TRIPLE INTEGRALS

Suppose we have a function $f(x, y, z)$ defined in a closed three-dimensional region Q. Now, we can divide Q into n sub-regions ΔQ_k of ΔV_k, $k = 1, 2, \ldots, n$. Let (a_k, b_k, c_k) be any point of ΔQ_k.

Hence, the sum can be given as

$$\sum_{k=1}^{n} f(a_k, b_k, c_k)\,\Delta V_k$$

Now, let us consider

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(a_k, b_k, c_k)\,\Delta V_k$$

where limit is taken such that n increases indefinitely and the largest linear dimensions of each ΔQ_k approaches zero.

If the limit exists, then the triple integral of $f(x, y, z)$ over region Q is denoted by

$$\iiint_Q f(x,y,z)dV$$

The limit of $f(x, y, z)$ is continuous in Q.

APPLICATIONS OF INTEGRALS

Integration is used in a wide variety of calculations ranging from the most fundamental to advance physical and mathematical calculations. In this section, we discuss the methods of integration to calculate the area of curves, length of the curves and the volume of revolution.

Area of Curve

Area bounded by the Cartesian curve $y = f(x)$, the x-axis and the ordinates $x = a$, $x = b$ is $\int_a^b y dx$.

As shown in Fig. 2, area bounded by the polar curve $r = f(\theta)$ and the radii vectors $\theta = \alpha, \beta$ is $\frac{1}{2}\int_\alpha^\beta r^2 d\theta$.

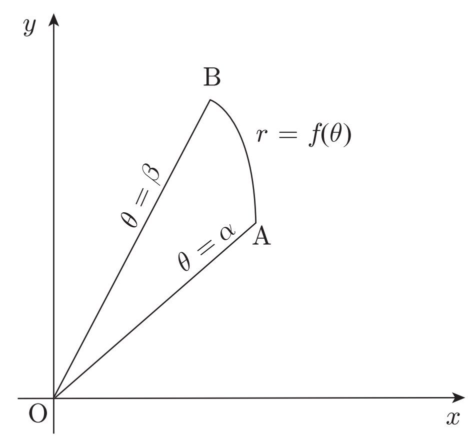


Figure 2 | Polar curve from A to B given by $r = f(\theta)$.

Length of Curve

Consider Fig. 3. The length of the arc of the curve $y = f(x)$ between the points where $x = a$ and $y = b$ is $\int_a^b \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}dx$.

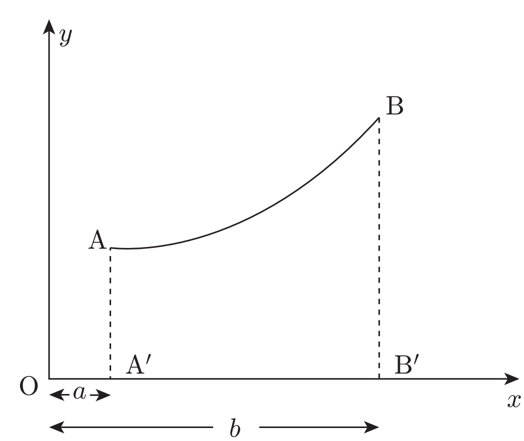


Figure 3 | Curve between A to B given by $y = f(x)$.

The length of the arc of the polar curve $r = f(\theta)$ between the points where $\theta = \alpha, \beta$ is $\int_{\alpha}^{\beta} \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]} d\theta.$

The length of the arc of the curve $x = f(t), y = g(t)$ between the points where $t = a$ and $t = b$ is

$$\int_{a}^{b} \sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]} dt.$$

Volumes of Revolution

The volume of the solid generated by the revolution about the x-axis, of the area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$, b is $\int_{a}^{b} \pi y^2 dx.$

Similarly, the volume of the solid generated by the revolution about the y-axis, of the area bounded by the curve $x = f(y)$, the y-axis and the abscissae $y = a$, b is $\int_{a}^{b} \pi\, x^2 dy.$

Consider Fig. 4. The volume of the solid generated by the revolution about any axis $A'B'$ of the area bounded by the curve AB, the axis and the two perpendiculars on the axes AA' and BB' is $\int_{OA'}^{OB'} \pi\,(CD)^2\, d(OD).$

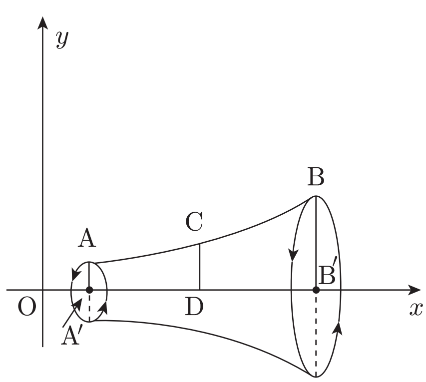


Figure 4 | Volume of solid generation by revolution of the curve AB about any axis $A'B'$.

Consider the polar curve shown in Fig. 5. The volume of the solid generated by the revolution of the area bounded by the curve $r = f(\theta)$ and the radii vectors $\theta = \alpha, \beta$ is given as follows:

(a) For $\theta = 0,$ (about the line OX)

$$\int_{\alpha}^{\beta} \frac{2\pi}{3} r^3 \sin\theta d\theta$$

(b) For $\theta = \frac{\pi}{2}$, (about the line OY)

$$\int_{\alpha}^{\beta} \frac{2\pi}{3} r^3 \cos\theta d\theta$$

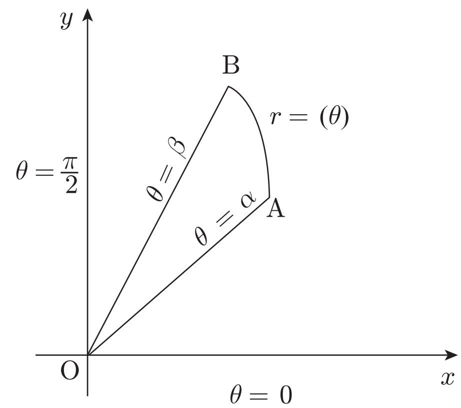


Figure 5 | Polar curve from A to B given by $r = f(\theta)$ for $\theta = \alpha$ and $\theta = \beta$.

FOURIER SERIES

Fourier series is a way to represent a wave-like function as a combination of sine and cosine waves. It decomposes any periodic function into the sum of a set of simple oscillating functions (sines and cosines). The Fourier series for the function $f(x)$ in the interval $\alpha < x < \alpha + 2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$a_0 = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \cos nx dx$$

$$b_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \sin nx dx$$

The values of a_0, a_n and b_n are known as Euler's formulae.

Conditions for Fourier Expansion

Fourier expansion can be performed on any function $f(x)$ if it fulfills the Dirichlet conditions. The Dirichlet conditions are given as follows:

1. $f(x)$ should be periodic, single-valued and finite.
2. $f(x)$ should have a finite number of discontinuities in any one period.
3. $f(x)$ should have a finite number of maxima and minima.

Fourier Expansion of Discontinuous Function

While deriving the values of a_0, a_n, b_n, we assumed $f(x)$ to be continuous. However, a function may be expressed as a Fourier transform even if the function has a finite number of points of finite discontinuity.

Let us say that we have a function $f(x)$ in the interval $\alpha < x < \alpha + 2\pi$ and $f(x)$ is defined by

$$f(x) = \phi_1(x), \quad \alpha < x < c$$
$$\phi_2(x), \quad c < x < \alpha + 2\pi$$

where c is the point of discontinuity.

Now, we can define the Euler's formulae as follows:

$$a_0 = \frac{1}{\pi}\left[\int_\alpha^c \phi_1(x)dx + \int_c^{\alpha+2\pi} \phi_2(x)dx\right]$$
$$a_n = \frac{1}{\pi}\left[\int_\alpha^c \phi_1(x)\cos nx dx + \int_c^{\alpha+2\pi} \phi_2(x)\cos nx dx\right]$$
$$b_n = \frac{1}{\pi}\left[\int_\alpha^c \phi_1(x)\sin nx dx + \int_c^{\alpha+2\pi} \phi_2(x)\sin nx dx\right]$$

At $x = c$, there is a finite jump in the graph of function. Both the limits, left-hand limit (i.e. $f(c - 0)$) and right-hand limit (i.e. $f(c + 0)$), exist and are different. At such a point, Fourier series gives the value of $f(x)$ as the arithmetic mean of these two limits. Hence, at $x = c$,

$$f(x) = \frac{1}{2}\left[f(c-0) + f(c+0)\right]$$

Change of Interval

Till now, we have talked about functions having periods of 2π. However, often the period of the function required to be expanded is some other interval (say $2c$). Then, the Fourier expansion is given as follows:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{c} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{c}$$

where

$$a_0 = \frac{1}{c}\int_\alpha^{\alpha+2c} f(x)dx$$
$$a_n = \frac{1}{c}\int_\alpha^{\alpha+2c} f(x)\cos\frac{n\pi x}{c}dx$$
$$b_n = \frac{1}{c}\int_\alpha^{\alpha+2c} f(x)\sin\frac{n\pi x}{c}dx$$

Fourier Series Expansion of Even and Odd Functions

We already know that a function $f(x)$ is said to be even if $f(-x) = f(x)$ and $f(x)$ is said to be even if $f(-x) = -f(x)$.

Now, a periodic function $f(x)$ defined in $(-c, c)$ can be represented by the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{c} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{c}$$

where

$$a_0 = \frac{1}{c}\int_{-c}^{c} f(x)dx$$
$$a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}dx$$
$$b_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}dx$$

Case 1:

When $f(x)$ is an even function,

$$a_0 = \frac{1}{c}\int_{-c}^{c} f(x)dx = \frac{2}{c}\int_0^c f(x)dx$$
$$a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}dx = \frac{2}{c}\int_0^c f(x)\cos\frac{n\pi x}{c}dx$$

However, since $f(x)\sin\frac{n\pi x}{c}$ is even, $b_n = \frac{1}{c}\int_{-c}^{c} f(x)$ $\sin\frac{n\pi x}{c}dx = 0$.

Thus, if a periodic function $f(x)$ is even, its Fourier expansion contains only cosine terms, a_0 and a_n.

Case 2:

When $f(x)$ is an odd function,

$$a_0 = \frac{1}{c}\int_{-c}^{c} f(x)dx = 0$$
$$a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}dx = 0$$

However,

$$b_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}dx = \frac{2}{c}\int_0^c f(x)\sin\frac{n\pi x}{c}dx.$$

Thus, if a periodic function $f(x)$ is odd, its Fourier expansion contains only sine terms and b_n.

Half Range Series

Sometimes, it is required to obtain a Fourier expansion of a function $f(x)$ for the range $(0, c)$, which is half the period of the Fourier series. The Fourier expansion of such a function of half the period, therefore, consists of sine or cosine terms only.

If $f(x)$ is required to be expanded as a sine series in $0 < x < c$, then we extend the function reflecting it in the origin, so that $f(x) = -f(-x)$. The extended function is odd in $(-c, c)$ and the expansion can be given as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{c}$$

where $b_n = \frac{2}{c}\int_0^c f(x) \sin \frac{n\pi x}{c} dx$

If $f(x)$ is required to be expanded as a cosine series in $0 < x < c$, we extend the function reflecting it in the y-axis, so that $f(x) = f(-x)$. The extended function is even in $(-c, c)$ and the expansion can be given as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{c}$$

where

$$a_0 = \frac{2}{c}\int_0^c f(x) dx$$

$$a_n = \frac{2}{c}\int_0^c f(x) \cos \frac{n\pi x}{c} dx$$

VECTORS

Vector is any quantity that has magnitude as well as direction. If we have two points A and B, then vector between A and B is denoted by $\overrightarrow{AB}$.

Position vector is a vector of any points, A, with respect to the origin, O. If A is given by the coordinates x, y and z.

$$\left|\overrightarrow{AP}\right| = \sqrt{x^2 + y^2 + z^2}$$

1. *Zero vector* is a vector whose initial and final points are same. Zero vectors are denoted by $\vec{0}$. They are also called null vectors.
2. *Unit vector* is a vector whose magnitude is unity or one. It is in the direction of given vector $\vec{A}$ and is denoted by $\hat{A}$.
3. *Equal vectors* are those which have the same magnitude and direction regardless of their initial points.

Addition of Vectors

According to triangle law of vector addition, as shown in Fig. 6,

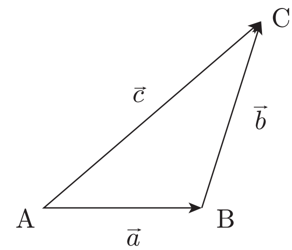


Figure 6 | Triangle law of vector addition.

$$\vec{c} = \vec{a} + \vec{b}$$

According to parallelogram law of vector addition, as shown in Fig. 7,

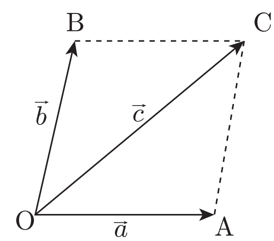


Figure 7 | Parallelogram law of vector addition.

$$\vec{c} = \vec{a} + \vec{b}$$

If we have two vectors represented by adjacent sides of parallelogram, then the sum of the two vectors in magnitude and direction is given by the diagonal of the parallelogram. This is known as parallelogram law of vector addition.

Some important properties of vector addition are:

1. If we have two vectors $\vec{a}$ and $\vec{b}$

$$\vec{a} + \vec{b} = \vec{b} + \vec{a}$$

2. If we have three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$

$$\left(\vec{a} + \vec{b}\right) + \vec{c} = \vec{a} + \left(\vec{b} + \vec{c}\right)$$

Multiplication of Vectors

1. **Multiplication of a vector with scalar:** Consider a vector $\vec{a}$ and a scalar quantity k. Then

$$|k\vec{a}| = |k||\vec{a}|$$

2. **Multiplication of a vector with another vector using dot product:** Dot product or scalar product of two vectors is given by

$$\vec{a} \cdot \vec{b} = |\vec{a}|\left|\vec{b}\right| \cos\theta$$

where $|\vec{a}|$ = magnitude of vector $\vec{a}$, $|\vec{b}|$ = magnitude of vector $\vec{b}$ and θ = angle between $\vec{a}$ and $\vec{b}, 0 \leq \theta \leq \pi$.

The result of dot product of two vectors is a scalar. Dot product is zero when both the vectors are perpendicular to each other. Dot product is maximum when both the vectors are in the same direction and is minimum when both the vectors are in the opposite direction.

Multiplication of Vectors Using Cross Product

The cross or vector product of two vectors is given by

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta\, \hat{n},$$

where $|\vec{a}|$ = magnitude of vector $\vec{a}$

$|\vec{b}|$ = magnitude of vector $\vec{b}$

θ = angle between $\vec{a}$ and $\vec{b}$, $0 \leq \theta \leq \pi$

$\hat{n}$ = Unit vector perpendicular to both $\vec{a}$ and $\vec{b}$.

The result of $\vec{a} \times \vec{b}$ is always a vector.

Cross product is zero if the vectors are in the same direction or in the opposite direction (i.e. $\theta = 0$ or 180°). Cross product is maximum if the angle between the two vectors is 90°, and it is minimum if the angle between the two vectors is 270°.

Some important laws of dot product are:

1. $A \cdot B = B \cdot A$
2. $A \cdot B + C = A \cdot B + A \cdot C$
3. $k(A \cdot B) = (kA) \cdot B = A \cdot (kB)$
4. $i \cdot i = j \cdot j = k \cdot k = 1,\ i \cdot j = j \cdot k = k \cdot i = 0$

Some important laws of cross product are:

1. $A \times B = -B \times A$
2. $A \times (B + C) = A \times B + A \times C$
3. $m(A \times B) = (ma) \times B = A \times (mB)$
4. $i \times i = j \times j = k \times k = 0,\ i \times j = k,\ j \times k = i,\ k \times i = j$
5. If $A = A_1 i + A_2 j + A_3 k$ and $B = B_1 i + B_2 j + B_3 k$, then

$$A \times B = \begin{vmatrix} i & j & k \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

Derivatives of Vector Functions

The derivative of vector $A(x)$ is defined as

$$\frac{dA}{dx} = \lim_{\Delta x \to 0} \frac{A(x + \Delta x) - A(x)}{\Delta x}$$

if the above limits exists.

If $A(x) = A_1(x)i + A_2(x)j + A_3(x)k$, then

$$\frac{dA}{dx} = \frac{dA_1}{dx} i + \frac{dA_2}{dx} j + \frac{dA_3}{dx} k$$

If $A(x, y, z) = A_1(x, y, z)i + A_2(x, y, z)j + A_3(x, y, z)k$, then

$$dA = \frac{dA}{dx} dx + \frac{dA}{dy} dy + \frac{dA}{dz} dz$$

$$\frac{d}{dy}(A \cdot B) = A \frac{dB}{dy} + \frac{dA}{dy} B$$

$$\frac{d}{dz}(A \times B) = A \times \frac{dB}{dz} + \frac{dA}{dz} \times B$$

A unit vector perpendicular to two given vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{c} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$$

Gradient of a Scalar Field

If we have a scalar function $a(x, y, z)$, then the gradient of this scalar function is a vector function which is defined by

$$\text{grad } \vec{a} = \nabla a = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

The gradient is basically defined as a vector of the same magnitude and direction as that of the maximum space rate of change of $\vec{a}$.

Divergence of a Vector

If we have a differentiable vector $\vec{A}(x, y, z)$, then divergence of vector $\vec{A}$ is given by

$$\text{div } \vec{A} = \nabla \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

where A_x, A_y and A_z are the components of vector $\vec{A}$.

Curl of a Vector

The curl of a continuously differentiable vector $\vec{A}$ is given by

$$\text{curl } \vec{A} = \nabla \vec{A} = \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z}\right) \times (A_x i + A_y j + A_z k)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) i + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) j + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) k$$

where A_x, A_y and A_z are the components of vector $\vec{A}$.

Thus, the curl of a vector $\vec{A}$ is defined as a vector function of space obtained by taking the vector product of $\vec{A}$.

Some important points of divergence and curl are:

1. $\nabla \cdot \nabla \vec{A} = \nabla^2 A = \dfrac{\partial^2 \vec{A}}{\partial x^2} + \dfrac{\partial^2 \vec{A}}{\partial y^2} + \dfrac{\partial^2 \vec{A}}{\partial z^2}$
2. $\nabla \times \nabla \vec{A} = 0$
3. $\nabla \cdot \nabla \times \vec{A} = 0$
4. $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) \times \nabla^2 \vec{A}$
5. $\nabla (\nabla \cdot \vec{A}) = \nabla \times (\nabla \times \vec{A}) + \nabla^2 \vec{A}$
6. $\nabla (\vec{A} + \vec{B}) = \nabla \cdot \vec{A} + \nabla \cdot \vec{B}$
7. $\nabla \times (\vec{A} + \vec{B}) = \nabla \times \vec{A} + \nabla \times \vec{B}$
8. $\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$
9. $\nabla \times (\vec{A} \times \vec{B}) = (B \cdot \nabla) A - B (\nabla \cdot A) - (A \nabla) B + A (\nabla B)$

Directional Derivative

The directional derivative of a multivariate differentiate function $f(x, y, z)$ is the rate at which the function changes at a point (x_0, y_0, z_0) in the direction of a vector v.

The directional derivative of a scalar function, $f(x) = f(x_1, x_2, \ldots, x_n)$ along a vector $v = (v_1, \ldots, v_n)$ is a function defined by the limit

$$\nabla_v f(x) = \lim_{h \to 0} \frac{f(x + h\hat{v}) - f(x)}{h}$$

If the function f is differentiable at x, then the directional derivative exists alone any vector v,

$$\nabla_v f(x) = \nabla f(x) \cdot \hat{v}$$

where ∇ on the right-hand side of the equation is the gradient and $\hat{v}$ is the unit vector given by $\hat{v} = \dfrac{v}{|v|}$.

The directional derivative is also often written as follows:

$$\frac{d}{dv} = \hat{v} \cdot \nabla = v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z}$$

Scalar Triple Product

The scalar triple product of three vectors is defined as the dot product of one of the vectors with the cross product of the other two vectors.

Thus, the scalar product of three vectors a, b and c is defined as

$$[a, b, c] = a \cdot (b \times c) = b \cdot (c \times a) = c \cdot (a \times b)$$

The volume of a parallelepiped whose sides are given by the vectors a, b and c, as shown in Fig. 8, can be calculated by the absolute value of the scalar triple product.

$$V_{\text{parallelepiped}} = |a \cdot (b \times c)|$$

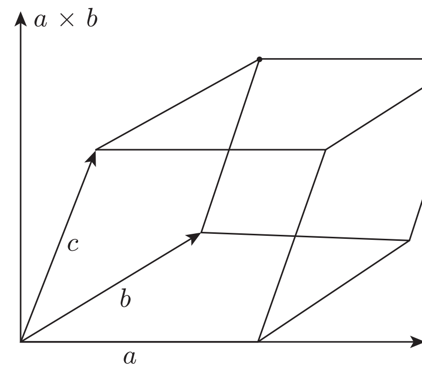


Figure 8 | Parallelepiped with sides given by the vectors a, b and c.

Vector Triple Product

The vector triple product of any three vectors is defined as the cross product of one vector with the cross product of the other two vectors.

Hence, if we have three vectors a, b and c, then the vector triple product is given by

$$a \times (b \times c) = b(a \cdot c) - c(a \cdot b)$$

LINE INTEGRALS

Let c be a curve from points $A(a_1, b_1)$ and $B(a_2, b_2)$ on the xy plane. Let $P(x, y)$ and $Q(x, y)$ be single-valued functions defined at all points of c. Now, c is divided into n parts and $(n - 1)$ points are chosen as $(x_1, y_1), (x_2, y_2), \ldots, (x_{n-1}, y_{n-1})$. Let us define $\Delta x_k = x_k - x_{k-1}$ and $\Delta y_k = y_k - y_{k-1}$, $k = 1, 2, \ldots, n$ where $(a_1, b_1) \equiv (x_0, y_0)$ and $(a_2, b_2) \equiv (x_n, y_n)$.

Let us say that points (α_k, β_k) are chosen so that they lie on the curve between points (x_{k-1}, y_{k-1}) and (x_k, y_k).

Now, consider the sum,

$$\sum_{k=1}^{n} \{P(\alpha_k, \beta_k)\, \Delta x_k + Q(\alpha_k, \beta_k)\, \Delta y_k\}$$

The limit of the sum as $n \to \infty$ is taken in such a way that all the quantities Δx_k and Δy_k approach zero, and if such limit exists, it is called a line integral along the curve C and is denoted by

$$\int_c [P(x,y)dx + Q(x,y)dy]$$

The limit exists if P and Q are continuous at all points of C. To understand better, refer to Fig. 9.

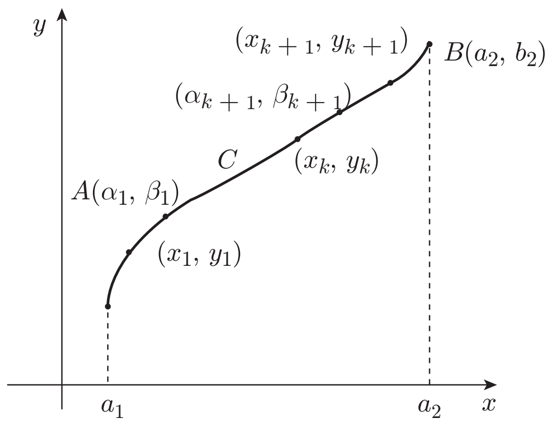


Figure 9 | Line integral.

In the same way, a line integral along a curve C in the three-dimensional space is given by

$$\int_c [A_1 dx + A_2 dy + A_3 dz]$$

where A_1, A_2 and A_3 are functions of x, y and z, respectively.

SURFACE INTEGRALS

Consider C to be a two-sided surface having a projection C' on the xy plane. The equation for C is $z = f(x, y)$, where f is a continuous single-valued function for all values of x and y.

Now, divide C' into n sub-regions of area ΔA_k where $k = 0, 1, 2, \ldots, n$ and join a vertical column on each of the corresponding sub-regions to intersect C in an area ΔC_p.

Suppose $g(x, y, z)$ is single valued and continuous for all values of C. Now, consider the sum

$$\sum_{k=1}^{n} g(\alpha_k, \beta_k, \gamma_k)\, \Delta c_p$$

where $(\alpha_k, \beta_k, \gamma_k)$ is any arbitrary point of ΔC_k. If the limit of this sum, $n \to \infty$ is in such a way that each $\Delta C_k \to 0$ exists, the resulting limit is called the surface integral of $g(x, y, z)$ over C.

The surface integral is denoted by

$$\iint_C g(x, y, z) \cdot ds$$

Figure 10 graphically represents surface integrals.

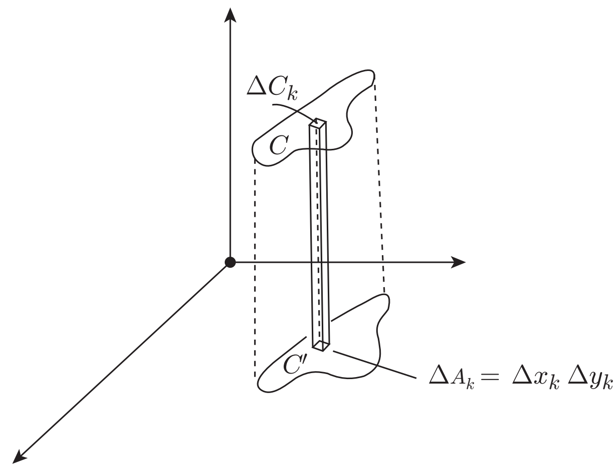


Figure 10 | Surface integrals.

STOKES' THEOREM

Let S be an open, two-sided surface bounded by a simple closed curve C. Also, if $\vec{V}$ is a single-valued, continuous function, then according to Stokes' theorem, "The line integral of the tangential component of a vector $\vec{V}$ taken around a simple closed curve C is equal to the surface integral of the normal component of the curl of $\vec{V}$ taken over any surface S having C as the boundary."

It is denoted by

$$\int_c A \cdot dr = \iint_s (\nabla \times A) \cdot n ds$$

GREEN'S THEOREM

Let S be a surface bounded by a simple closed curve C. Let $f_1(x, y)$ and $f_2(x, y)$ be continuous functions and $\frac{\partial f_1}{\partial x}$ and $\frac{\partial f_2}{\partial y}$ be continuous partial derivatives in S, then according to Green's theorem,

$$\iint_s \left(\frac{\partial f_1}{\partial x} - \frac{\partial f_2}{\partial y} \right) dx dy = \oint_c (f_2 dx + f_1 dy)$$

GAUSS DIVERGENCE THEOREM

Let S be a closed surface bounding a region of volume V. Assuming the outward drawn normal to the surface as positive normal and considering α, β and γ as the angles which this normal makes with x, y and z axes, respectively.

Also, if A_1, A_2 and A_3 are continuous and have continuous partial derivatives in the region, then

$$\iiint_v \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) dV = \iint_s (A_1 \cos\alpha + A_2 \cos\beta + A_3 \cos\gamma) dS$$

$$\Rightarrow \iiint_v \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) dV = \iint_s (A_1 dydz + A_2 dzdx + A_3 dxdy)$$

In vector form, with $A = A_1 i + A_2 j + A_3 k$ and $\hat{n} = \cos\alpha i + \cos\beta j + \cos\gamma k$,

$$\iiint_v \nabla \cdot A dV = \iint_s A \cdot \hat{n} dS$$

Divergence theorem states that the surface integral of the normal components of a vector $\vec{A}$ taken over a closed surface is equal to the integral of the divergence of $\vec{A}$ taken over the volume enclosed by the surface.

SOLVED EXAMPLES

1. What is the value of $\lim_{x \to 0} \frac{\sin\left[\frac{4}{3}x\right]}{x}$?

Solution: We have

$$\lim_{x \to 0} \frac{\sin\left[\frac{4}{3}x\right]}{x}$$

$$= \lim_{\frac{4}{3}x \to 0} \frac{4}{3} \frac{\sin\left[\frac{4}{3}x\right]}{\frac{4}{3}x}$$

$$= \frac{4}{3} \lim_{\frac{4}{3}x \to 0} \frac{\sin\left[\frac{4}{3}x\right]}{\frac{4}{3}x}$$

$$= \frac{4}{3} \times 1 = \frac{4}{3}$$

2. What is the value of $\lim_{x \to 0} \frac{x^3 - 6x^2 + 11x - 6}{x^2 - 6x + 8}$?

Solution: When $x \to 2$, $\frac{x^3 - 6x^2 + 11x - 6}{x^2 - 6x + 8} = \frac{0}{0}$

Hence, we apply L'Hospital's rule,

$$\lim_{x \to 2} \frac{3x^2 - 12x + 11}{2x - 6} = \frac{3(2)^2 - 12(2) + 11}{2(2) - 6}$$

$$= \frac{12 - 24 + 11}{-2} = \frac{-1}{-2} = \frac{1}{2}$$

3. What is the value of $\lim_{x \to \infty} \frac{x + \sin x}{x - x\cos x}$?

Solution: We have

$$\lim_{x \to \infty} \frac{x + \sin x}{x - x\cos x} = \lim_{x \to \infty} \frac{1 + \frac{\sin x}{x}}{1 + \frac{\cos x}{x}} = \frac{1 - \lim_{x \to \infty} \frac{\sin x}{x}}{1 + \lim_{x \to \infty} \frac{\cos x}{x}}$$

$$= \frac{1 - 0}{1 + 0} = 1$$

4. What is the value of $\lim_{x \to 0} \frac{1 - \cos 2x}{x}$?

Solution: We have

$$\lim_{x \to 0} \frac{1 - \cos 2x}{x}$$

$$= \lim_{x \to 0} \frac{2\sin^2 x}{x}$$

$$= 2 \lim_{x \to 0} \left(\frac{\sin x}{x}\right) \cdot \sin x = 2\left(\lim_{x \to 0} \frac{\sin x}{x}\right)\left(\lim_{x \to 0} \sin x\right)$$

$$= 2(1)(0) = 0$$

5. If a function is given by

$$f(x) = \begin{cases} \frac{\sin x}{x} + \cos x, & x \neq 0 \\ 2, & x = 0 \end{cases}$$

Find out whether or not $f(x)$ is continuous at $x = 0$.

Solution: We have

L.H.L. at $x = 0$

$$= \lim_{x \to 0} f(x) = \lim_{h \to 0} f(0 - h) = \lim_{h \to 0} f(-h)$$

$$= \lim_{h \to 0} \left[\frac{\sin(-h)}{-h} + \cos(-h)\right] = 1 + 1 = 2$$

R.H.L. at $x = 0$

$$= \lim_{x \to 0} f(x) = \lim_{h \to 0} f(0 + h) = \lim_{h \to 0} f(h)$$

$$= \lim_{h \to 0} \left[\frac{\sin h}{h} + \cos h\right] = 1 + 1 = 2$$

Also, we know that $f(0) = 2$. Thus, $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$.

Hence, $f(x)$ is continuous at $x = 0$.

6. Discuss the continuity of the function $f(x)$ at $x = 1/2$, where

$$f(x) = \begin{cases} 1/2^{-x}, & 0 \le x < 1/2 \\ 1, & x = 1/2 \\ 3/2^{-x}, & 1/2 < x \le 1 \end{cases}$$

Solution: We have

LHL at $x = 1/2$

$$= \lim_{x \to 1/2^-} f(x) = \lim_{x \to 1/2} (1/2 - x)$$

$$= -1/2 - 1/2 = 0$$

RHL at $x = 1/2$

$$= \lim_{x \to 1/2^+} f(x) = \lim_{x \to 1/2} (3/2 - x)$$

$$= 3/2 - 1/2 = 1$$

Since, $\lim_{x \to \frac{1}{2}^-} f(x) \ne \lim_{x \to \frac{1}{2}^+} f(x)$

Hence, $f(x)$ not continuous at $x = \frac{1}{2}$.

7. Discuss the continuity of $f(x) = 2x - |x|$ at $x = 0$.

Solution: We have

$$f(x) = 2x - |x| = \begin{cases} 2x - x, & \text{if} \quad x \ge 0 \\ 2x - (-x), & \text{if} \quad x < 0 \end{cases}$$

$$\Rightarrow f(x) = \begin{cases} x, & \text{if} \quad x \ge 0 \\ 3x, & \text{if} \quad x < 0 \end{cases}$$

Now,

L.H.L. at $x = 0$

$$= \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 3x = 3 \times 0 = 0$$

R.H.L. at $x = 0$

$$= \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x = 0$$

and $f(0) = 0$

$$= \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} x \, f(x) = f(0)$$

So, $f(x)$ is continuous at $x = 0$.

8. For what value of λ is the function $f(x)$ continuous at $x = 3$?

$$f(x) = \begin{cases} \dfrac{x^2 - 9}{x - 3}, & x \ne 3 \\ \lambda, & x = 3 \end{cases}$$

Solution: Since $f(x)$ is continuous at $x = 3$,

$$\lim_{x \to 3} f(x) = f(3)$$

$$\lim_{x \to 3} f(x) = \lambda$$

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lambda$$

$$\lim_{x \to 3} \frac{(x+3)(x-3)}{(x-3)} = \lambda$$

$$\lim_{x \to 3} (x + 3) = \lambda$$

$$3 + 3 = \lambda$$

$$\lambda = 6$$

9. Discuss the differentiability of the function

$$f(x) = \begin{cases} x - 1, & \text{if } x < 2 \\ 2x - 3, & \text{if } x \ge 2 \end{cases}$$

Solution: At $x = 2$

$$(\text{L.H.D. at } x = 2) = \lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2}$$

$$= \lim_{x \to 2} \frac{(x-1) - (4-3)}{x - 2}$$

$$= \lim_{x \to 2} \frac{x - 2}{x - 2} = \lim_{x \to 2} 1 = 1$$

$$(\text{R.H.D. at } x = 2) = \lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2}$$

$$= \lim_{x \to 2} \frac{(2x-3) - (4-3)}{x - 2}$$

$$= \lim_{x \to 2} \frac{2(x-2)}{x - 2} = \lim_{x \to 2} 2 = 2$$

Therefore, L.H.D. $\ne$ R.H.D.

Hence, $f(x)$ is not differentiable at $x = 2$.

10. Discuss the differentiability of $f(x) = x|x|$ at $x = 0$.

Solution: We have

$$f(x) = x|x| = \begin{cases} x^2, & x \ge 0 \\ -x^2, & x < 0 \end{cases}$$

$$(\text{L.H.D. at } x = 0) = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \to 0} \frac{-x^2 - 0}{x - 0} = \lim_{x \to 0} (-x) = 0$$

$$(\text{R.H.D. at } x = 0) = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \to 0} \frac{x^2 - 0}{x - 0} = \lim_{x \to 0} x = 0$$

Therefore, (L.H.D. at $x = 0$) = (R.H.D. at $x = 0$)

Hence, $f(x)$ is differentiable at $x = 0$.

11. Discuss the applicability of Rolle's theorem for the function $f(x) = x^2 - 5x + 6$ on the interval [2, 3].

Solution: We know that

(i) $f(x)$ is continuous on [2, 3]

(ii) $f(x)$ is differentiable on [2, 3]

[$\because$ a polynomial function is differentiable everywhere]

(iii) $f(2) = (2)^2 - 5(2) + 6 = 0$

$f(3) = (3)^2 - 5(3) + 6 = 0$

Thus, $f(2) = f(3)$.

Hence, Rolle's theorem is applicable. Therefore, there exists a value $c \in (2,3)$ such that $f'(c) = 0$.

We have

$$f(x) = x^2 - 5x + 6 \quad \Rightarrow \quad f'(x) = 2x - 5$$

$$f'(x) = 2x - 5 = 0 \quad \Rightarrow \quad x = 2.5$$

Thus, $c = 2.5 \in (2,3)$ such that $f'(c) = 0$.

Hence, Rolle's theorem is verified.

12. Discuss the applicability of Rolle's theorem for $f(x) = |x|$ on $[-1, 1]$.

Solution: We have

$$f(x) = \begin{cases} -x, & \text{when } -1 \le x < 0 \\ x, & \text{when } 0 \le x \le 6 \end{cases}$$

Function $f(x)$ is continuous and differentiable at all points $x < 0$ and $x > 0$, since it is a polynomial function.

However, we have to check for continuity and differentiability at $x = 0$.

Now, $\lim_{x \to 0^-} f(x) = \lim_{x \to 0} (-x) = 0$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0} x = 0$

$f(0) = 0$

$\therefore \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$

Thus, $f(x)$ is continuous at $x = 0$ and hence continuous on $[-1, 1]$.

Checking for differentiability,

$$\text{(L.H.D. at } x = 0) = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \to 0} \frac{-x - 0}{x - 0} = \lim_{x \to 0} (-1) = -1$$

$$\text{(R.H.D. at } x = 0) = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \to 0} \frac{x - 0}{x} = \lim_{x \to 0} 1 = 1$$

(L.H.D. at $x = 0$) $\neq$ (R.H.D. at $x = 0$)

Hence, $f(x)$ is not differentiable at $x = 0 \in (-1, 1)$.

Thus, Rolle's theorem is not applicable.

13. Verify Rolle's theorem for the function $f(x) = \sin x + \cos x - 1$ on $[0, \pi/2]$.

Solution: As $\sin x$ and $\cos x$ are continuous and differentiable everywhere, $f(x) = \sin x + \cos x - 1$ is continuous on $[0, \pi/2]$ and differentiable on $(0, \pi/2)$.

Now, $f(0) = \sin 0 + \cos 0 - 1 = 0 + 1 - 1$

$= 0$

$f(\pi/2) = \sin \pi/2 + \cos \pi/2 - 1 = 1 + 0 - 1$

$= 0$

$\therefore f(0) = f(\pi/2)$

Thus, $f(x)$ satisfies all the conditions of Rolle's theorem. Therefore, Rolle's theorem is applicable, i.e. there exists $c \in (0, \pi/2)$ such that $f'(c) = 0$.

Now,

$$f(x) = \sin x + \cos x - 1 \quad \Rightarrow f'(x) = \cos x - \sin x$$

Also, $f'(x) = 0$

$\therefore f'(x) = \cos x - \sin x = 0$

$\Rightarrow \sin x = \cos x$

$\Rightarrow \tan x = 1$

$\Rightarrow x = \pi/4$

Thus, $c = \pi/4 \in (0, \pi/2)$ such that $f'(c) = 0$.

14. Verify Lagrange's mean value theorem for $f(x) = x(x-2)$ on [1, 3].

Solution: We have

$$f(x) = x(x-2) = x^2 - 2x$$

We know that a polynomial function is continuous and differentiable everywhere. So, $f(x)$ is continuous on [1, 3] and differentiable on (1, 3).

Hence, $f(x)$ satisfies both the conditions of Langrage's mean value theorem on [1, 3], and hence there exists at least one real number $c \in (1, 3)$ such that

$$f'(c) = \frac{f(3) - f(1)}{3 - 1}$$

Now, $f(x) = x^2 - 2x$

$\Rightarrow f'(x) = 2x - 2$

$f(3) = 9 - 6 = 3$ and $f(1) = 1 - 2 = -1$

$$f'(x) = \frac{f(3) - f(1)}{3 - 1}$$

$$\Rightarrow 2x - 2 = \frac{3 - (-1)}{3 - 1}$$

$\Rightarrow 2x - 2 = 2 \Rightarrow x = 2$

Thus, $c = 2 \in (1, 3)$ such that $f'(c) = \dfrac{f(3) - f(1)}{3 - 1}$.

Hence, Lagrange's mean value theorem is verified for $f(x)$ on $[1, 3]$.

15. Verify Lagrange's mean value theorem for $f(x) = 2\sin x + \sin 2x$ on $[0, \pi]$.

Solution: $\sin x$ and $\sin 2x$ are continuous and differentiable everywhere, therefore $f(x)$ is continuous on $[0, \pi]$ and differentiable on $(0, \pi)$.

Thus, $f(x)$ satisfies both the conditions of Lagrange's mean value theorem.

There exists at least one $c \in (0, \pi)$ such that

$$f'(c) = \frac{f(\pi) - f(0)}{\pi - 0}$$

$f(x) = 2\sin x + \sin 2x$

$f'(x) = 2\cos x + 2\cos 2x$

$f(0)$ and $f(\pi) = 2\sin \pi + \sin 2\pi = 0$

$$\therefore\ f'(x) = \frac{f(\pi) - f(0)}{\pi - 0}$$

$$\Rightarrow \quad 2\cos x + 2\cos 2x = \frac{0 - 0}{\pi - 0}$$

$\Rightarrow \quad 2\cos x + 2\cos 2x = 0$

$\Rightarrow \quad \cos x + \cos 2x = 0$

$\Rightarrow \quad \cos 2x = -\cos x$

$\Rightarrow \quad \cos 2x = \cos(\pi - x)$

$\Rightarrow \quad 2x = \pi - x$

$\Rightarrow \quad 3x = \pi \quad \Rightarrow x = \pi/3$

Thus, $c = \pi/3 \in (0, \pi)$ such that $f'(c) = \dfrac{f(\pi) - f(0)}{\pi - 0}$.

Hence, Lagrange's mean value theorem is verified.

16. Differentiate the function $f(x) = \tan^{-1}\left(\dfrac{\sqrt{1 + x^2} + 1}{x}\right)$.

Solution: Let $y = \tan^{-1}\left(\dfrac{\sqrt{1 + x^2} + 1}{x}\right)$, put $x = \tan\theta$, i.e. $\theta = \tan^{-1} x$.

Now, after substituting the value, we get

$$y = \tan^{-1}\left(\frac{\sqrt{1 + \tan^2\theta} + 1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{\sec\theta + 1}{\tan\theta}\right) = \tan^{-1}\left[\frac{\frac{1}{\cos\theta} + 1}{\frac{\sin\theta}{\cos\theta}}\right]$$

$$= \tan^{-1}\left(\frac{1 + \cos\theta}{\sin\theta}\right) = \tan^{-1}\left[\frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right]$$

$$= \tan^{-1}\left(\cot\frac{\theta}{2}\right) = \tan^{-1}\left[\tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right]$$

$$= \frac{\pi}{2} - \frac{\theta}{2} = \frac{\pi}{2} - \frac{1}{2}\tan^{-1} x$$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = 0 - \frac{1}{2} \cdot \frac{1}{1 + x^2} = \frac{-1}{2(1 + x^2)}$$

17. Differentiate the function $f(x) = \dfrac{e^x + \sin x}{1 + \log x}$

Solution: We have

$$f(x) = \frac{e^x + \sin x}{1 + \log x}$$

$$f'(x) = \frac{(1 + \log x)\frac{d}{dx}(e^x + \sin x) - (e^x + \sin x)\frac{d}{dx}(1 + \log x)}{(1 + \log x)^2}$$

$$= \frac{(1 + \log x)(e^x + \cos x) - (e^x + \sin x)\left[0 + \frac{1}{x}\right]}{(1 + \log x)^2}$$

$$= \frac{(1 + \log x)(e^x + \cos x) - \frac{e^x + \sin x}{x}}{(1 + \log x)^2}$$

18. Differentiate the function $f(x) = x^{x^x}$.

Solution: Let $y = x^{x^x}$, then

$$y = ex^x \cdot \log x$$

On differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = e^{x\log x^x}\frac{d}{dx}(x^x \cdot \log x)$$

$$\Rightarrow \frac{dy}{dx} = x^{x^x}\frac{d}{dx}(e^{x\log x} \cdot \log x)$$

$$\Rightarrow \frac{dy}{dx} = x^{x^x}\left[\log x \cdot e^{x\log x}\frac{d}{dx}(x\log x) + e^{x\log x}\cdot\frac{d}{dx}(\log x)\right]$$

$$= x^{x^x}\left[\log x \cdot e^{x\log x}\frac{d}{dx}(x\log x) + e^{x\log x}\cdot\frac{1}{x}\right]$$

$$= x^{x^x}\left[\log x \cdot x^x\left(x\cdot\frac{1}{x}+\log x\right) + x^x\cdot\frac{1}{x}\right]$$

$$= x^{x^x}\left[x^x(1+\log x)\cdot\log x + \frac{x^x}{x}\right]$$

$$= x^{x^x}\cdot x^x\left[(1+\log x)\cdot\log x + \frac{1}{x}\right]$$

19. If $f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ \sin x, & 0 \le x \le \pi \end{cases}$, prove that $f(x) = \frac{1}{\pi} + \frac{\sin x}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2-1}$. Hence, show that $\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots - \infty = \frac{1}{4}(\pi-2)$.

Solution: Let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Then,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0dx + \int_0^{\pi}\sin xdx\right] = \frac{2}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0dx + \int_0^{\pi}\sin x\cos nxdx\right]$$

$$= \frac{1}{2\pi}\int_0^{\pi}\left[\sin(n+1)x - \sin(n-1)x\right]dx$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right] \quad (n \ne 1)$$

$$= \frac{1}{2\pi}\left\{\frac{1-(-1)^{n+1}}{n+1} - \frac{(-1)^{n-1}-1}{n-1}\right\} = 0, \quad \text{when } n \text{ is odd}$$

$$\text{and } -\frac{2}{\pi(n^2-1)}, \quad \text{when } n \text{ is even}$$

When $n = 1$,

$$a_1 = \frac{1}{\pi}\int_0^{\pi}\sin x\cos xdx = \frac{1}{2\pi}\int_0^{\pi}\sin 2xdx$$

$$= \frac{1}{2\pi}\left|-\frac{\cos 2x}{2}\right|_0^{\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0dx + \int_0^{\pi}\sin x\sin nxdx\right]$$

$$= \frac{1}{2\pi}\int_0^{\pi}\left[\cos\overline{n-1}x - \cos\overline{n+1}x\right]dx$$

$$= \frac{1}{2\pi}\left[\frac{\sin\overline{n-1}x}{n-1} - \frac{\sin\overline{n+1}x}{n+1}\right]_0^{\pi} = 0 \quad (n \ne 1)$$

When $n = 1$,

$$b_1 = \frac{1}{\pi}\int_0^{\pi}\sin x\sin xdx$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi}(1-\cos 2x)dx\right]$$

$$= \frac{1}{2\pi}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi} = \frac{1}{2}$$

Hence,

$$f(x) = \frac{1}{\pi} - \frac{2}{\pi}\left[\frac{\cos 2x}{2^2-1} + \frac{\cos 4x}{4^2-1} + \frac{\cos 6x}{6^2-1} + \cdots\right] + \frac{1}{2}\sin x \qquad (1)$$

Putting $x = \frac{\pi}{2}$ in Eq. (1), we get 1

$$= \frac{1}{\pi} - \frac{2}{\pi}\left(-\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} - \frac{1}{5\cdot 7} + \cdots\infty\right) + \frac{1}{2}$$

Hence, $\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots\infty = \frac{1}{4}(\pi - 2)$.

20. Prove that the function $f(x) = x^3 - 6x^2 + 12x - 9$ is strictly increasing on R.

Solution: We have

$$f(x) = x^3 - 6x^2 + 12x - 9, \quad x \in R$$

Differentiating with respect to x, we get

$$f'(x) = 3x^2 - 12x + 12 = 3(x^2 - 4x + 4)$$

$$= 3(x-2)^2 \ge 3 \quad (\because (x-2)^2 \ge 0 \text{ for all } x \in R)$$

$\Rightarrow f'(x) > 0$ for all $x \in R$

$\Rightarrow f(x)$ is strictly increasing function for all $x \in R$.

21. Find the intervals in which the function $f(x) = x^4 - \frac{x^3}{3}$ is decreasing.

Solution: We have

$$f(x) = x^4 - \frac{x^3}{3}$$

$$\Rightarrow \quad f'(x) = 4x^3 - x^2 = x^2(4x-1)$$

For $f(x)$ to be decreasing, we have $f'(x) < 0$

$$x^2(4x-1) < 0$$

$$\Rightarrow \quad (4x-1) < 0 \qquad (\because x^2 > 0)$$

$$\Rightarrow \quad 4x < 1 \quad \Rightarrow x < \frac{1}{4}$$

$$\Rightarrow \quad x \in \left(-\infty, \frac{1}{4}\right)$$

Hence, $f(x)$ is decreasing on $\left(-\infty, \frac{1}{4}\right)$.

22. Find the points of local maxima and the corresponding maximum values of the function,

$$f(x) = 2x^3 - 21x^2 + 36x$$

Solution: We have

$$f(x) = 2x^3 - 21x^2 + 36x$$

$$f'(x) = 6x^2 - 42x + 36$$

For local maximum, we have $f'(x) = 0$

$$\Rightarrow \quad 6x^2 - 42x + 36 = 0 \quad \Rightarrow (x-1)(x-6) = 0$$

$$\Rightarrow x = 1, 6$$

Thus, $x = 1$ and $x = 6$ are the possible points of local maxima or minima.

Now, $f''(x) = 12x - 42$

At $x = 1$, we have

$$f''(1) = 12 - 42 = -30 < 0$$

Hence, $x = 1$ is a point of local maximum.

The local maximum value is $f(1) = 2 - 21 + 36 - 20 = -3$

At x= 6, we have

$$f''(6) = 12(6) - 42 = 30 > 0$$

Hence, $x = 6$ is a point of local maximum.

23. Find the points of local maxima or minima for the function,

$$f(x) = x^3 + x$$

Solution: We have

$$f(x) = x^3 + x$$

$$f'(x) = 3x^3 + 1$$

For a maximum or minimum, we have

$$f'(x) = 0 \quad \Rightarrow \quad 3x^2 + 1 = 0 \quad \Rightarrow \quad x = \pm\frac{i}{\sqrt{3}}$$

This gives the imaginary values of x, hence $f'(x) \neq 0$ for any real value of x.

Hence, $f(x)$ does not have a maximum or minimum.

24. Find the absolute maximum and minimum values of $f(x) = \sin x + \frac{1}{2}\cos 2x$ in $\left[0, \frac{\pi}{2}\right]$.

Solution: We have

$$f(x) = \sin x + \frac{1}{2}\cos 2x \quad \text{in } \left[0, \frac{\pi}{2}\right]$$

Differentiating with respect to x, we get

$$f'(x) = \cos x - \sin 2x$$

For absolute maximum and absolute minimum,

$$f'(x) = 0$$

$$\Rightarrow \cos x - 2\sin x \cos x = 0$$

$$\Rightarrow \cos x(1 - 2\sin x) = 0$$

$$\Rightarrow \cos x = 0 \quad \text{or} \quad \sin x = \frac{1}{4}$$

$$x = \frac{\pi}{2} \text{ or } \frac{\pi}{6}$$

Now, $f\left(\frac{\pi}{6}\right) = \sin\frac{\pi}{6} + \frac{1}{2}\cos\frac{\pi}{3} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

$$f\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} + \frac{1}{2}\cos\pi = 1 - \frac{1}{2} = \frac{1}{2}$$

$$f(0) = \sin 0 + \frac{1}{2}\cos 0 = 0 + \frac{1}{2} = \frac{1}{2}$$

The absolute maximum value $= 3/4$.

The absolute minimum value $= 1/2$.

25. Derive the Taylor's series expansion of $\frac{\sin x}{x - \pi}$ at $x = \pi$.

Solution: The Taylor's series expansion of $f(x)$ around $x = \pi$ is

$$f(x) = f(\pi) + \frac{x-\pi}{\lfloor 1}f'(x) + \frac{(x-\pi)^2}{\lfloor 2}f''(\pi) + \cdots$$

Now, $f(\pi) = \lim_{x \to \pi} \frac{\sin x}{x - \pi} = \frac{0}{0}$

Hence, we apply L'Hospital's rule,

$$\lim_{x \to \pi} \frac{\cos x}{1} = -1$$

Similarly by using L'Hospital's rule, we can show that

$$f'(\pi) = 0$$
$$f''(\pi) = -1/6$$

So, the expansion is $f(x) = -1 + (-1/6)(x - \pi)^2 + \cdots$

Thus, $f(x) = -1 - \frac{(x - \pi^2)}{\lfloor 3} + \cdots$

26. Expand $e^{\sin x}$ by Maclaurin's series up to the term containing x^4.

Solution: We have

$$f(x) = e^{\sin x}$$
$$f'(x) = e^{\sin x} \cos x \cdot f(x) \cdot \cos x$$
$$f''(x) = f'(x) \cos x - f(x) \sin x \quad f''(0) = 1$$
$$f'''(x) = f''(x) \cos x - 2f'(x) \sin x$$
$$-f(x) \cos x, f'''(0) = 0$$
$$f''''(x) = f'''(x) \cos x - 3f'(x) \sin x$$
$$-3f'(x) \cos x \cdot f(x) \sin x,$$
$$f''''(0) = 0$$

and so on.

Substituting the values of $f(0), f'(0)$, etc. in the Maclaurin's series, we get

$$e^{(\sin x)} = 1 + x \cdot 1 + \frac{x^2 \cdot 1}{2!} + \frac{x^3 \cdot 0}{3!} + \frac{x^4 \cdot (-3)}{4!} + \cdots$$
$$= 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \cdots$$

27. Expand $\log \tan[(\pi/4) + x]$ in ascending power of x till x^5.

Solution: Using Taylor's theorem, we know that

$$f(x+h) = f(h) + x f'(h) + \frac{x^2}{2!} f''(h) + \frac{x^3}{3!} f'''(h) + \cdots \quad (1)$$

If $f(x+h) = \log \tan[x + (\pi/4)]$

then $f(x) = \log \tan x$ and $f(h) = \log \tan(h)$ (2)

Differentiating Eq. (2) successively with respect to h, we get

$$f'(h) = \frac{\sec^2 h}{\tan h} = 2 \operatorname{cosec} 2h$$
$$f''(h) = -4 \operatorname{cosec} 2h \cot 2h$$
$$f'''(h) = -8 - [\operatorname{cosec} 2h \cot^2 2h + 5 \operatorname{cosec}^3 2h]$$
$$f''''(h) = -16 [\operatorname{cosec} 2h \cot^3 2h + 5 \operatorname{cosec}^3 2h \cot 2h]$$
$$f'''''(h) = 32 [\operatorname{cosec} 2h \cot^4 2h + 3 \cot^2 2h \operatorname{cosec}^3 2h + 5 \operatorname{cosec}^5 2h + 15 \operatorname{cosec}^3 2h \cot^2 2h]$$

Now, substituting the value of $h = \pi/4$, we get

$$f(\pi/4) = 0$$
$$f'(\pi/4) = 2$$
$$f''(\pi/4) = 0$$
$$f'''(\pi/4) = 8$$
$$f''''(\pi/4) = 0$$
$$f'''''(\pi/4) = 160$$

Putting these values in Eq. (1), we have

$$f(x+h) = 0 + 2x - \frac{x^2}{2!}(0) + \frac{x^3}{3!} \times 8 + \frac{x^4}{4!} \times (0) + \frac{x^5}{5!} \times 160 = 2x + \frac{4}{3}x^3 + \frac{4}{3}x^5 + \cdots$$

28. Evaluate $\int \frac{2x}{\sqrt{1 - x^2 - x^4}} dx$

Solution: Let $x^2 = t$

Then, $d(x^2) = dt \Rightarrow 2x \cdot dx = dt \Rightarrow dx = \frac{dt}{2x}$

Therefore,

$$f(x) = \int \frac{2x}{\sqrt{1 - x^2 - x^4}} dx = \int \frac{dt}{\sqrt{1 - t - t^2}}$$
$$= \int \frac{dt}{\sqrt{-(t^2 + t - 1)}}$$
$$= \int \frac{dt}{\sqrt{-\left(t^2 + t + \frac{1}{4} - \frac{1}{4} - 1\right)}}$$
$$= \int \frac{dt}{\sqrt{-\left[\left(t + \frac{1}{2}\right)^2 - \frac{5}{4}\right]}}$$
$$= \int \frac{dt}{\sqrt{5/4 - \left(t + \frac{1}{2}\right)^2}} = \int \frac{dt}{\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - \left(t + \frac{1}{2}\right)^2}}$$
$$f(x) = \sin^{-1}\left[\frac{t + 1/2}{\sqrt{5}/2}\right] + C$$
$$= \sin^{-1}\left(\frac{2t + 1}{\sqrt{5}}\right) + C$$

Substituting $t = x^2$,

$$f(x) = \sin^{-1}\left(\frac{2x^2 + 1}{\sqrt{5}}\right) + C$$

29. Find the Fourier series expansion of $f(x) = 2x - x^2$ in (0, 3) and hence deduce that $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots - \infty = \frac{\pi}{12}$.

Solution: The required series is of the form,

$$2x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where $l = 3/2$.

Then

$$a_0 = \frac{1}{l}\int_0^{2l}(2x - x^2)dx = \frac{2}{3}\left|x^2 - \frac{x^3}{3}\right|_0^3 = 0$$

$$a_n = \frac{1}{l}\int_0^{2l}(2x - x^2)\cos\frac{n\pi x}{l}dx$$

$$= \frac{2}{3}\int_0^3 (2x - x^2)\cos\frac{2n\pi x}{3}dx$$

$$= \frac{2}{3}\left[(2x - x^2)\frac{\sin 2n\pi x}{2n\pi/3} - (2 - 2x)\frac{-\cos 2n\pi x/3}{(2n\pi/3)^2} + (-2)\frac{-\sin 2n\pi x/3}{(2n\pi/3)^3}\right]_0^3$$

$$= \frac{2}{3}\cdot\frac{9}{4n^2\pi^2}\left[(2 - 6)\cos 2n\pi - 2\right] = -\frac{9}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_0^{2l}(2x - x^2)\sin\frac{n\pi x}{l}dx$$

$$= \frac{2}{3}\int_0^3 (2x - x^2)\sin\frac{2n\pi x}{3}dx$$

$$= \frac{2}{3}\left[(2x - x^2)\frac{-\cos 2n\pi x/3}{2n\pi/3} - (2 - 2x)\frac{-\sin 2n\pi x/3}{(2n\pi/3)^2} + (-2)\frac{\cos 2n\pi x/3}{(2n\pi/3)^3}\right]_0^3$$

$$= \frac{2}{3}\left\{-\frac{6}{n^2\pi^2}\cos 2n\pi - \frac{27}{4n^3\pi^3}(\cos 2n\pi - 1)\right\}$$

$$= \frac{3}{n\pi}$$

Substituting the values of a_0, a_n, b_n in Eq. (1), we get

$$2x - x^2 = -\sum_{n=1}^{\infty}\frac{9}{n^2\pi^2}\cos\frac{2n\pi x}{3} + \sum_{n=1}^{\infty}\frac{3}{n\pi}\sin\frac{2n\pi x}{3}$$

30. If $f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ \sin x, & 0 \le x \le \pi \end{cases}$, prove that $f(x) = \frac{1}{\pi} + \frac{\sin x}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2 - 1}$. Hence, show that $\frac{1}{1\cdot 3} - \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots - \infty = \frac{1}{4}(\pi - 2)$

Solution: Let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Then,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0dx + \int_0^{\pi}\sin x dx\right] = \frac{2}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0dx + \int_0^{\pi}\sin x\cos nx dx\right]$$

$$= \frac{1}{2\pi}\int_0^{\pi}\left[\sin(n+1)x - \sin(n-1)x\right]dx$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right] \quad (n \ne 1)$$

$$= \frac{1}{2\pi}\left\{\frac{1 - (-1)^{n+1}}{n+1} - \frac{(-1)^{n-1} - 1}{n-1}\right\} = 0,$$

when n is odd

and $-\frac{2}{\pi(n^2 - 1)}$, when n is even.

When $n = 1$,

$$a_1 = \frac{1}{\pi}\int_0^{\pi}\sin x\cos x dx = \frac{1}{2\pi}\int_0^{\pi}\sin 2x dx$$

$$= \frac{1}{2\pi}\left|-\frac{\cos 2x}{2}\right|_0^{\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{\pi}\left|\int_{-\pi}^{0} 0dx + \int_0^{\pi}\sin x\sin nx dx\right|$$

$$= \frac{1}{2\pi}\int_0^{\pi}\left[\cos\overline{n-1}x - \cos\overline{n+1}x\right]dx$$

$$= \frac{1}{2\pi}\left[\frac{\sin\overline{n-1}x}{n-1} - \frac{\sin\overline{n+1}x}{n+1}\right]_0^{\pi} = 0 \quad (n \neq 1)$$

When $n = 1$,

$$b_1 = \frac{1}{\pi}\int_0^{\pi}\sin x \sin x\, dx = \frac{1}{2\pi}\left[\int_0^{\pi}(1-\cos 2x)\,dx\right]$$

$$= \frac{1}{2\pi}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi} = \frac{1}{2}$$

Hence, $$f(x) = \frac{1}{\pi} - \frac{2}{\pi}\left[\frac{\cos 2x}{2^2-1} + \frac{\cos 4x}{4^2-1} + \frac{\cos 6x}{6^2-1} + \cdots\right] + \frac{1}{2}\sin x \quad (1)$$

Putting $x = \frac{\pi}{2}$ in Eq. (1), we get $1 = \frac{1}{\pi} - \frac{2}{\pi}\left(-\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} - \frac{1}{5\cdot 7} + \cdots\infty\right) + \frac{1}{2}$

Hence, $$\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots\infty = \frac{1}{4}(\pi - 2)$$

31. Evaluate $\int x\sin^{-1} dx$.

Solution: We have

$$f(x) = \int x\sin^{-1}x\cdot dx$$

$$= \sin^{-1}x\cdot\frac{x^2}{2} - \int\frac{1}{\sqrt{1-x^2}}\cdot\frac{x^2}{2}\cdot dx$$

$$= \frac{x^2}{2}\cdot\sin^{-1}x + \frac{1}{2}\int\frac{-x^2}{\sqrt{1-x^2}}\cdot dx = \frac{x^2}{2}\cdot\sin^{-1}x + \frac{1}{2}\int\frac{1-x^2-1}{\sqrt{1-x^2}}\cdot dx$$

$$= \frac{x^2}{2}\cdot\sin^{-1}x + \frac{1}{2}\left[\int\frac{1-x^2}{\sqrt{1-x^2}}\cdot dx - \int\frac{1}{\sqrt{1-x^2}}\cdot dx\right]$$

$$= \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\left[\int\sqrt{1-x^2}\,dx - \int\frac{1}{\sqrt{1-x^2}}\,dx\right]$$

$$= \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\left[\left(\frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right) - \sin^{-1}x\right] + \text{C}$$

$$= \frac{1}{2}x^2\sin^{-1}x + \frac{1}{4}x\sqrt{1-x^2} - \frac{1}{4}\sin^{-1}x + \text{C}$$

32. Evaluate $\int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}\,dx.$

Solution: Let

$$f(x) = \int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}\,dx \quad (1)$$

Since, $\int_0^a f(x)\cdot dx = \int_0^a f(a-x)dx$

$$f(x) = \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{x}}\cdot dx \quad (2)$$

Adding Eqs. (1) and (2), we get

$$2f(x) = \int_0^a \frac{\sqrt{x}+\sqrt{a-x}}{\sqrt{x}+\sqrt{a-x}}\cdot dx$$

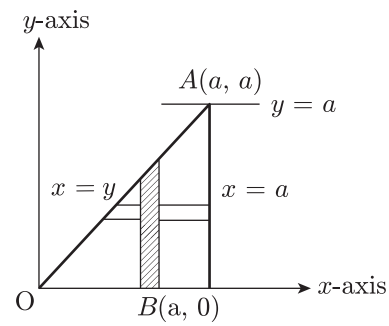
$$2f(x) = \int_0^a dx$$

$$2f(x) = a$$

$$f(x) = \frac{a}{2}$$

33. Change the order of integration in $\int_0^a\int_y^a \frac{x\cdot dx dy}{x^2+y^2}$ and evaluate the same.

Solution: From the limit of integration, it is clear that region of integration is bounded by $x = y$, $x = a$, $y = 0$ and $y = a$. Hence, region of integration can be formed as follows:



Hence, it is clear that region of integration is given by ΔOAB and divided into horizontal strips. For changing the order of integration, we divide the region into vertical strips.

Now, the new limits become 0 to x for y and 0 to a for x.

$$\Rightarrow \int_0^a\int_y^a \frac{x}{x^2+y^2}\,dx\,dy = \int_0^a\int_0^x \frac{x}{x^2+y^2}\,dy\,dx$$

$$= \int_0^a x\left[\frac{1}{x}\tan^{-1}\frac{y}{x}\right]_0^x dx$$

$$= \int_0^a \frac{\pi}{4}\,dx = \frac{\pi}{4}[x]_0^a = \frac{\pi a}{4}$$

34. Find the area of the triangle whose vertices are $A(3, -1, 2)$, $B(1, -1, -3)$ and $C(4, -3, 1)$.

Solution: Let $\vec{a}, \vec{b}, \vec{c}$ be the positive vectors of points $A(3, -1, 2)$, $B(1, -1, -3)$ and $C(4, -3, 1)$, respectively.

Then, $\vec{a} = 3\hat{i} - \hat{j} + 2\hat{k}$, $\vec{b} = \hat{i} - \hat{j} - 3\hat{k}$ and $\vec{c} = 4\hat{i} - 3\hat{j} + \hat{k}$.

We have

Area of triangle $= \dfrac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$

Now, $\overrightarrow{AB} = \vec{b} - \vec{a} = \hat{i} - \hat{j} - 3\hat{k} - (3\hat{i} - \hat{j} + 2\hat{k}) = 2\hat{i} - 5\hat{k}$

$$\overrightarrow{AC} = \vec{c} - \vec{a} = 4\hat{i} - 3\hat{j} + \hat{k} - (3\hat{i} - \hat{j} + 2\hat{k}) = \hat{i} - 2\hat{j} - \hat{k}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 0 & -5 \\ 1 & -2 & -1 \end{vmatrix}$$

$$= (0 - 10)\hat{i} - (2 + 5)\hat{j} + (4 - 0)\hat{k}$$

$$= -10\hat{i} - 7\hat{j} + 4\hat{k}$$

$$\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \sqrt{(-10)^2 + (-7)^2 + (4)^2} = \sqrt{165}$$

Area of $\Delta ABC = \dfrac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \dfrac{1}{2}\sqrt{165}$

35. Prove that $\displaystyle\int_1^2\int_3^4 (xy + e^y)dy \cdot dx = \int_3^4\int_1^2 (xy + e^y)dx \cdot dy$

Solution: We have

$$\text{L.H.S.} = \int_1^2\int_3^4 (xy + e^y)dy \cdot dx = \int_1^2 \left[\frac{xy^2}{2} + e^y\right]_3^4 dx$$

$$= \int_1^2 \left(8x + e^4 - \frac{9}{2}x - e^3\right)dx$$

$$= \int_1^2 \left[\frac{7}{2}x + e^4 - e^3\right] \cdot dx = \left[\frac{7}{4}x^2 + (e^4 - e^3)x\right]_1^2$$

$$= 7 + 2(e^4 - e^3) - \frac{7}{4} - (e^4 - e^3)$$

$$= \frac{21}{4} + e^4 - e^3$$

Now, R.H.S. $= \displaystyle\int_3^4\int_1^2 (xy + e^y)\, dx \cdot dy$

$$= \int_3^4 \left[\frac{x^2y}{2} + e^y \cdot x\right]_1^2 \cdot dy$$

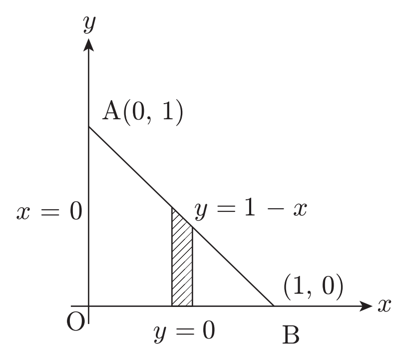
$$\int_3^4 \left[2y + 2e^y - \frac{y}{2} - e^y\right] \cdot dy = \int_3^4 \left(\frac{3}{2}y + e^y\right) \cdot dy$$

$$= \left(\frac{3}{4}y^2 + e^y\right)_3^4 = 12 + e^4 - \frac{27}{4} - e^3 = \frac{21}{4} + e^4 - e^3$$

Therefore, L.H.S. = R.H.S. Hence proved.

36. Evaluate $\displaystyle\iint_R e^{2x+3y} \cdot dxdy$ over the triangle bounded by $x = 0, y = 0$ and $x + y = 1$.

Solution: The region R of integration is ΔAOB. Here, x varies from 0 to 1 and y varies from x-axis up to the line $x + y = 1$, i.e. from 0 to $1 - x$.



The region R can be expressed as follows:

$$0 \le x \le 1 \quad \text{and} \quad 0 \le y \le 1 - x$$

Therefore, $\displaystyle\iint_R e^{2x+3y} \cdot dxdy = \int_0^1\int_0^{1-x} e^{2x+3y} \cdot dydx$

$$= \int_0^1 \left[\frac{1}{3} \cdot e^{2x+3y}\right]_0^{1-x} \cdot dx = \frac{1}{3}\int_0^1 (e^{3-x} - e^{2x}) \cdot dx$$

$$= \frac{1}{3}\left[-e^{3-x} - \frac{1}{2}e^{2x}\right]_0^1 = -\frac{1}{3}\left[e^2 + \frac{1}{2}e^2 - e^3 - \frac{1}{2}\right]$$

$$= -\frac{1}{3}\left[-e^2(e-1) + \frac{1}{2}(e^2 - 1)\right]$$

$$= -\frac{1}{6}(e-1)\left[2e^2 - (e+1)\right] = \frac{1}{6}(e-1)(2e^2 - e - 1)$$

$$= \frac{1}{6}(e-1)(e-1)(2e+1) = \frac{1}{6}(e-1)^2(2e+1)$$

37. Expand $f(x) = \begin{cases} \dfrac{1}{4} - x, & \text{if } 0 < x < \dfrac{1}{2} \\ x - \dfrac{3}{4}, & \text{if } \dfrac{1}{2} < x < 1 \end{cases}$ as the Fourier series of sine terms.