CHAPTER 1

ENGINEERING MECHANICS

Engineering mechanics is the branch of scientific analysis that describes and predicts the behavior of a stationary or moving body under the action of forces. In general, the subject is subdivided into three branches: rigid-body mechanics, deformable-body mechanics, and fluid mechanics. Present context is concerned with the mechanics of rigid-bodies that forms the foundation of mechanical devices. It is based on the assumption that the bodies are perfectly rigid. This is studied in two parts:

1. **Statics** *Statics* is concerned with mechanics of stationary systems. It requires the study of equilibrium stationary structures under forces and torque systems.
2. **Dynamics** *Dynamics* is concerned with the systems variant with time, and deals with the motion resulting from unbalanced force or torque systems. Dynamics is subdivided into two branches:
   (a) **Kinematics** *Kinematics* describes the motion of bodies without reference to the forces which either cause the motion or are generated as a result of motion. The subject is also referred to as the '*geometry of motion*'.
   (b) **Kinetics** *Kinetics* deals with the motion of rigid bodies under the action of forces.

1.1 FORCE

Force is a vector quantity which tends to change the state of a body. It means force is capable to bring a static body into motion or a moving body into static position. The study of mechanics encounters various types of force systems. The forces meeting at one point constitute a *concurrent force* system. The forces lying in one plane constitute a *coplanar force* system.

The SI unit of force is Newton (N), defined as the force acting on mass of 1 kg which produces an acceleration of 1 m/s^2.

1.1.1 Characteristics of a Force

Let a force $\vec{F}$ acts on a rigid body placed on a rough horizontal plane. Depending upon the *magnitude* of $\vec{F}$, the body can start moving in a straight line, if the *line of action* of $\vec{F}$ passes through the *center of gravity* of the body. This motion is called *translation*. If line of action does not pass through the center of gravity of the body, the force will also result into *rotation* of the body. Thus, a force is characterized by its magnitude, line of action, direction and point of application.

1.1.2 Resolutions of a Force

Force is a vector quantity, therefore, has its resolved components in given directions, which are called *resolutions*. Engineering problems frequently need resolution of a force in orthogonal directions. Consider a force F in a x–y plane at an angle θ with the x-axis [Fig. 1.1].

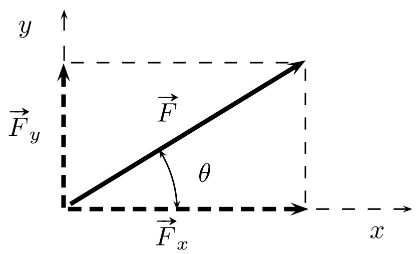

Figure 1.1 Resolution of a force.

Magnitudes of the resolved parts of force F along x and y directions are the given by

$$F_x = F\cos\theta$$
$$F_y = F\cos\left(\frac{\pi}{2} - \theta\right)$$

Thus, the resolved part of a given force in a given direction is equal to the magnitude of the force multiplied with the cosine of the angle between the line of action of the force and the direction.

Using the resolved parts, a force can be presented in vector form:

$$\vec{F} = F_x\hat{i} + F_y\hat{j}$$

where $\hat{i}$ and $\hat{j}$ are the unit vectors in x and y directions, respectively. Magnitude of force $\vec{F}$ can be found as

$$F = \sqrt{F_x^2 + F_y^2}$$

The concept of resolved components is used to add two or more forces by summing their x and y components:

$$R_x = \sum F_x$$
$$R_y = \sum F_y$$

where R_x and R_y are the resolved components of the resultant force expressed as

$$\vec{R} = R_x\hat{i} + R_y\hat{j}$$

1.2 MOMENT OF A FORCE

1.2.1 Definition

Moment of a force about a point or axis is the measure of the tendency of the force to cause a body to rotate about the point or axis. It is quantified by the product of the force and the perpendicular distance of its line of action from the point. This perpendicular distance is called *arm of the moment.*

Consider a force $\vec{F}$ acting on a rigid point. Moment of this force can be determined about a point O situated at distance $\vec{r}$ from line of action [Fig. 1.2].

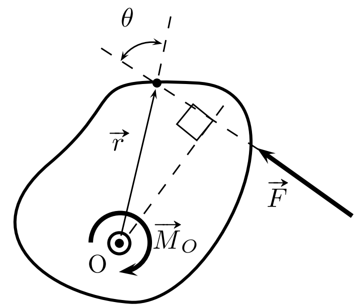

Figure 1.2 Moment of a force.

The moment $\vec{M}_O$ of the force about point O is defined as the cross product of force vector and distance vector:

$$\vec{M}_O = \vec{r} \times \vec{F} \tag{1.1}$$

The direction of $\vec{M}_O$ is determined by the right hand rule. Magnitude of the moment is given by

$$M_O = rF\sin\theta \tag{1.2}$$

where θ is the angle between $\vec{r}$ and $\vec{F}$.

The curl or sense of rotation can always be determined by observing in which direction the force would “orbit” about the fulcrum point O. The point is referred only for a two dimensional case, however, the moment always acts about an axis perpendicular to the plane containing $\vec{F}$ and $\vec{r}$, and this axis intersects the plane at the point O.

Eq. (1.2) indicates that a force will not contribute a moment about a specified axis if line of action of the force is parallel to the axis ($\theta = 0$).

1.2.2 Resultant Moment of a System of Forces

Let a system of forces acts upon a rigid body. Resultant moment of the forces about a point is determined by the vector addition of the moments of individual forces about that point:

$$\vec{M}_R = \sum(\vec{r} \times \vec{F})$$

1.2.3 Varignon's Theorem

According to *Varignon's theorem*[1], the algebraic sum of moments of several concurrent forces about any point is equal to the moments of their resultant about the same point.

Varignon's theorem can be stated alternatively as the moment of a force about any point equal to the sum of moments of its components about that point.

For the system of coplanar concurrent forces shown in Fig. 1.3, the Varignon's theorem is written as

$$Fr = F_x x + F_y y$$

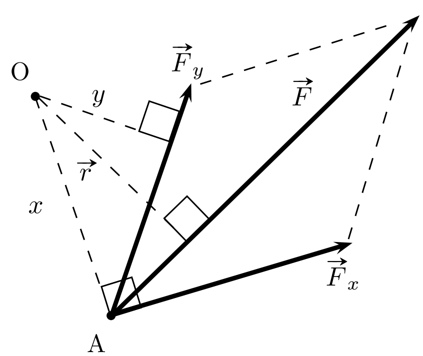

Figure 1.3 Varignon's theorem.

1.2.4 Principle of Moments

The *principle of moments* is a corollary derived from the Varignon's theorem, which states that if a system of coplanar forces is in equilibrium, then the algebraic sum of their moments about any point in their plane is zero.

1.2.5 Moment of a Couple

Two equal but opposite parallel forces having different lines of action form a *couple*[2]. The resultant force of the two forces in any direction is zero. However, the only effect is to produce a tendency to rotate a body upon which the couple act because sum of the moments of the two forces about a given point is not zero.

Let two forces $\vec{F}$ and $-\vec{F}$ be situated at distances $\vec{r}_1$ and $\vec{r}_2$ from a point O [Fig. 1.4].

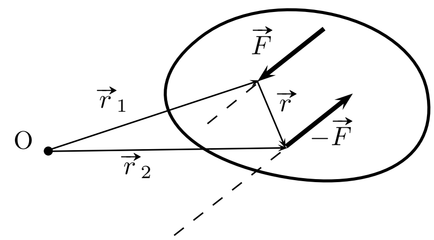

Figure 1.4 Moment of a couple.

The moment of the couple $\vec{M}$ is given by

$$\begin{aligned}\vec{M} &= \vec{r}_1 \times \vec{F} + \vec{r}_2 \times \left(-\vec{F}\right)\\ &= (\vec{r}_1 - \vec{r}_2) \times \vec{F}\\ &= \vec{r} \times \vec{F} \qquad (1.3)\end{aligned}$$

where $\vec{r}$ $(= \vec{r}_1 - \vec{r}_2)$ is the distance vector between the lines of action of the parallel forces. This vector is called *arm of the couple*. Direction of $\vec{M}$ is a vector perpendicular to $\vec{r}$ and force $\vec{F}$.

Equation (1.3) shows that *moment of the couple* is equal to the vector product of either force of the couple with the arm of the couple. The moment of a couple is independent of r_1 or r_2 vectors, therefore, point O can be chosen arbitrarily. It means that moment of a couple is a *free vector*, unlike the moment of a force which requires a definite axis.

1.3 EQUIVALENT SYSTEM OF A FORCE

A force tends to cause translation and rotation of a body. This depends upon the magnitude, direction, and line of action of the force with respect to the center of gravity of the body. A body can be subjected to a system of forces. The problem is generally simplified by determining an equivalent system of resultant force and moment that can produce the same effect of translation and rotation with respect to any point on the body.

Consider a body subjected to a force $\vec{F}$ at point P. The force is to be moved to another point O without changing the effect on the body [Fig. 1.5].

This can be done by applying equal and opposite forces $\vec{F}$ and $-\vec{F}$ at point O. Thus, the original force $\vec{F}$

[1] Pierre Varignon (1654–1722) was a French mathematician. He was a friend of Newton, Leibniz, and the Bernoulli family. His principal contributions were to graphic statics and mechanics. Varignon's theorem is a statement in Euclidean geometry by him that was first published in 1731.

[2] Moment is created by single force, while a couple is created by equal and opposite forces. Interestingly, a single force acting on a body creates a reaction in opposite direction from the body, thus constitutes a couple.

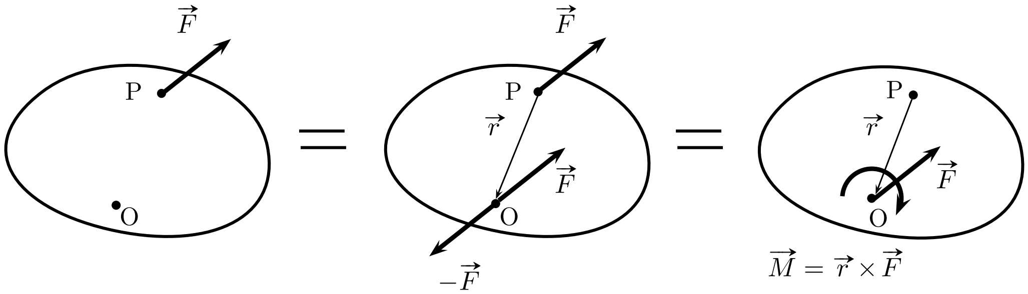

Figure 1.5 Equivalent system of a force.

at point A and force $-\vec{F}$ at point O form a couple whose moment depends upon the distance vector $\vec{r}$ between point O and P, like a *free vector*. Thus, an equivalent system of a force $\vec{F}$ acting at point A is found at point O.

1.4 SINGLE RESULTANT FORCE

Consider a situation when a rigid body is subjected to a system of forces and couple moments [Fig. 1.6]. The system of forces and moments can be reduced to a resultant force $\vec{F}_R$ acting at point O, and resultant moment $\vec{M}_R$ by the vector sum of the respective quantities:

$$\vec{F}_R = \sum \vec{F}$$
$$\vec{M}_R = \sum \vec{M}$$

As a special case, if $\vec{M}_R$ and $\vec{F}_R$ are perpendicular to each other, the situation can be further simplified by replacing $\vec{F}_R$ and $\vec{M}_R$ at point O by a single force $\vec{F}_R$ acting at a distance d from point O. The distance d is given by following expression:

$$d = \frac{M_R}{F_R}$$

This effect is the reverse of determining an equivalent force of a system. This observation can be applied in the following special cases:

1. *Concurrent Force Systems* The forces meeting at one point constitute a *concurrent force* system. Thus, there is no resultant couple moment, and the resultant force acts at a specific point O only.
2. *Coplanar Force Systems* The forces lying in one plane constitute a *coplanar force* system. Such a system can be replaced by the resultant coplanar force F acting at a point O, and the resultant moment M_R along an axis passing through point O and normal to the plane of forces. This can be further simplified by a resultant force F_R acting at a distance $d = M_R/F_R$ from point O.

1.5 EQUILIBRIUM OF RIGID BODIES

The concept of equilibrium of rigid bodies is derived from the *Newton's first law of motion*, which states that if the resultant force acting on a particle is zero, the particle will remain at rest (if originally at rest) or will move with constant speed in a straight line (if originally in motion). Thus, a body is considered in *equilibrium* when its condition (motion or rest) is unaffected by the forces acting on it. For example, a body moving with a constant acceleration caused by applied force is said to be in equilibrium.

1.5.1 General Condition

The necessary and sufficient conditions for complete equilibrium of a rigid body under a force system are as follows:

1. For any system of forces keeping a body in equilibrium, the algebraic sum of forces, in any direction is zero:
$$\sum \vec{F} = 0 \qquad (1.4)$$
2. For any system of forces keeping a body in equilibrium, the algebraic sum of the moments of all the forces about any point in their plane is zero:
$$\sum \vec{M} = 0 \qquad (1.5)$$

These are the fundamental equations of statics, which are essentially used in determining the unknown forces and reactions acting on a body under equilibrium. In this reference, a problem is called *statically determinate* if the number of unknown reactions is equal to the

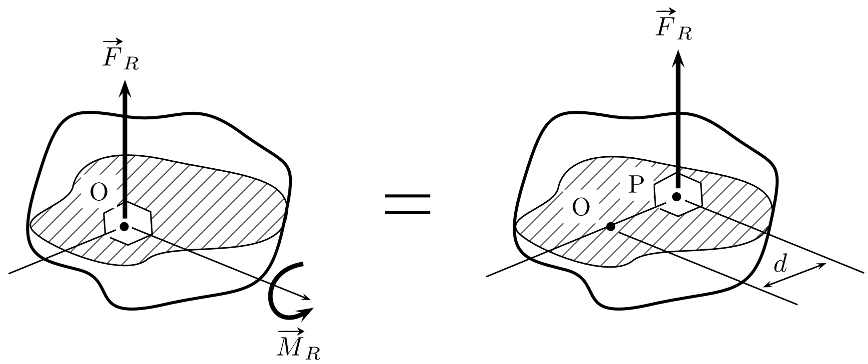

Figure 1.6 | Single resultant force.

number of equations of equilibrium. The problem is *statically indeterminate* if the number of unknown reactions is less than the number of equations of equilibrium.

1.5.2 Free Body Diagrams

Two bodies in contact exert forces on the other. One of these force is called *action*, and the other is called *reaction*. The concept of free body is derived from the *Newton's third law of motion* which states that action and reaction are always equal and opposite, and when bodies are smooth, they are normal to the surfaces in contact.

Equilibrium of a body can be examined using Eqs. (1.4)–(1.5). This requires knowledge of all the forces acting on a body. This is achieved by drawing the body's free body diagram. A diagram showing the forces acting on a body, together with reactions at the supports but not showing the supports is called a *free body diagram*. A body so isolated from its supports or surrounding is called a *free body*. Thus, a free body diagram shows all active and reactive forces acting on the body.

For example, consider a body resting on a surface [Fig. 1.7]. Its weight W acts downward which creates a normal reaction $\vec{R}_n$ at the surface. If a force $\vec{F}$ is applied to move the body in the horizontal direction, the surface exerts a friction force $\vec{F}'$ that acts opposite to it. The resultant of $\vec{R}_n$ and $\vec{F}'$ is R.

In the free body diagram, the weight W and force $\vec{F}$ are to be included along with the resultant reaction $\vec{R}$.

Internal forces of a body always occur in equal but opposite collinear pairs, therefore, their net effect on the body is zero. Thus, internal forces are not drawn in free body diagrams.

Weight of a body is the resultant of the gravity forces acting on the particles that constitute the body. The point of application of this resultant force is known as the center of gravity. Weight is an external force, thus, it is included in free body diagrams.

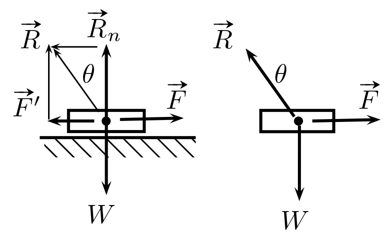

Figure 1.7 | Free body diagram.

1.5.3 Support Reactions

Knowledge of support reactions is necessary for drawing free body diagram of a body to examine its equilibrium using Eqs. (1.4)–(1.5). As a general rule, a support can prevent translation of a body in the given direction by exerting a reaction force on the body in the opposite direction. Likewise, a support can prevent the rotation of a body in a given direction by exerting a couple moment on the body in the opposite reaction. The force and couple moment are the reactions exerted by a support on a supported body.

The following are the three kinds of supports that offer different types of reactions [Fig. 1.8]:

1. *Roller Support* A roller support prevents the body from translation in the vertical direction because the roller can exert a reaction force along the common normal at tangent point.

2. *Hinged Support* A hinged or pin support does not offer resistance against rotation. Thus, it offers both horizontal and vertical reactions, but does not exert a couple moment on the supported body.

3. *Fixed Support* The most restrictive way to support a body is using a fixed support because it prevents both translation and rotation of a supported body. Thus, a fixed support offers all the three elements of reactions; horizontal and vertical reactions and moment.

(a) Roller support

(b) Hinged support

(c) Fixed support

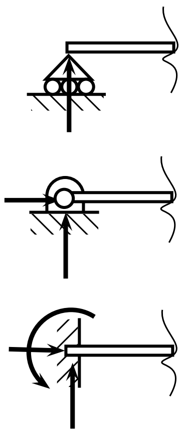

Figure 1.8 | Support reactions.

1.5.4 Equilibrium of Three Coplanar Forces

Using the general condition of equilibrium, the condition of equilibrium of three coplanar forces can be stated as follows:

1. If three coplanar forces acting upon a rigid body under equilibrium, they must either meet in a point or be all parallel.
2. If three forces are in equilibrium, they must be coplanar.

1.5.5 Triangle Law of Forces

The problem of equilibrium of three coplanar forces can be represented in triangular fashion. This is known as the *law of triangle of forces* which states that if three forces acting upon a particle can be presented in the magnitude and direction by the sides of a triangle taken in order, the forces will be in equilibrium.

In converse way, if three forces acting upon a particle in equilibrium, they can be represented in magnitude and direction by the sides of any triangle which is drawn so as to have its sides respectively parallel to the directions of the forces.

Consider three forces $\vec{F}_1$, $\vec{F}_2$, $\vec{F}_3$ acting on a particle or rigid body in equilibrium. The law of triangle of force is represented in Fig. 1.9 where these forces form a triangle.

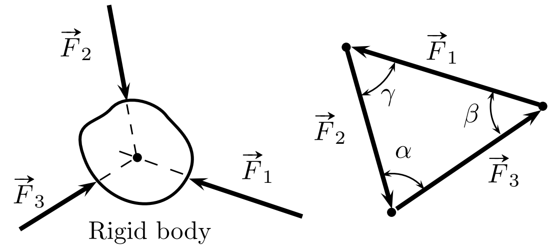

Figure 1.9 | Law of triangle of forces.

The law of triangle of forces is equivalent to the vector sum of the forces; the net force and moments acting on a particle is zero, therefore, particle is in equilibrium.

1.5.6 Lami's Theorem

In statics, *Lami's theorem*[3] is an equation that relates the magnitudes of three coplanar, concurrent and non-collinear forces, that keep a body in static equilibrium. The theorem states that if three forces acting at a point are in equilibrium, each force is proportional to the sine of the angle between the other two forces.

Consider three forces F_1, F_2, F_3 acting on a particle or rigid body making angles α, β, γ with each other [Fig. 1.10].

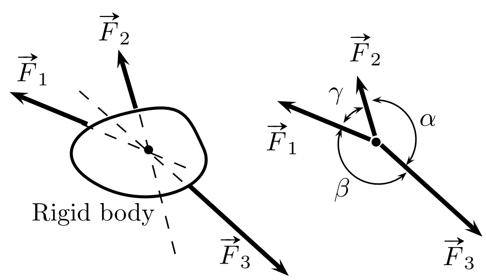

Figure 1.10 | Lami's theorem.

According to Lami's theorem, the particle shall be in equilibrium if

$$\frac{F_1}{\sin\alpha} = \frac{F_2}{\sin\beta} = \frac{F_3}{\sin\gamma} \tag{1.6}$$

The angle between the force vectors is taken when all the three vectors are emerging from the particle.

[3]Bernard Lamy (1640-1715) was a French Oratorian mathematician and theologian. His best known work is the parallelogram of forces (1679).

1.6 STRUCTURAL ANALYSIS

1.6.1 Trusses and Frames

A structure can consists two types of members:

1. *Trusses* A *truss* is an articulated structure composed of straight members arranged and connected in a such a way that they transmit primarily axial forces. For this, a truss is made up of several slender bars, called *members*, joined together at their ends by hinges or rivets. The bar members, therefore, act as two-force members which can either be in tension or in compression; there can be no transverse force in a member of a truss [Fig. 1.11].

   For the purpose of calculations, the joints (nodes) are supposed to be hinged or pin-jointed. A truss is designed to carry loads at the nodes, otherwise, truss members can be subjected to lateral loads.

   A perfect truss is composed of least number of members to prevent distortion of its shape when loaded. If the number of nodes in a perfect truss is n, then the minimum number of members is $2n-3$.

2. *Frame* A *frame* consists of members which can be subjected to a transverse load in addition to the axial load. Thus, members carry loads at points other than nodes. If load is applied at a point other than a joint, the member is subjected to bending also; and in such a case, the force in the member is not purely axial. To find the forces in the members subjected to bending, the equilibrium of each member is considered separately by constructing its free body diagram.

1.6.2 Assumptions

To determine the axial forces developed in the truss members, following assumptions are made:

1. Each truss is composed of rigid members, all lying in one plane.

2. Forces are transmitted from one member to another through smooth pins fitted in the members.

3. All the loads are applied at the joints.

4. Weight of the members are neglected because they are small in comparison with the loads.

The effect of axial forces acting at the joints of a member is shown by marking arrows over the member, according to the direction of the forces [Fig. 1.11].

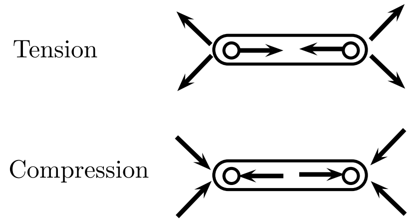

Figure 1.11 Sign of forces in members.

A member can be subjected to two types of axial forces: tension (arrow directed away from joint) or compression (arrow directed toward joint).

1.6.3 Method of Joints

A plane truss or frame can be subjected only to a coplanar force system. Therefore, any point on a member of the plane truss can be subjected to coplanar and concurrent systems only. This condition of concurrency of force system follows from the equilibrium of forces at a given point on a truss. The *method of joints* is based on these observations. This method takes one point at a time and analyzes it for equilibrium. At every node, the forces must be along the members at that joint and must satisfy the necessary conditions of equilibrium:

$$\sum \vec{F} = 0$$
$$\sum \vec{M} = 0$$

The sum of moments can be examined only at the points of application of the support reactions.

1.6.4 Zero-Force Members

Truss analysis using the method of joints is greatly simplified with the knowledge of zero-force members in the truss. Special situations of forces in truss members are explained as follows [Fig. 1.12]:

(a) If two members not in the same straight line meet at a point which does not carry any load, the force in each member is zero.

(b) If two members in the same straight line meet at a point, they carry equal and opposite forces.

(c) If three members meet at a joint which does not carry any load, and two members are in same line, then the force in third member will be zero.

(d) If four members meet at a point at which there is no load with two of the members in straight lines, then forces in members aligned in the same lines will be equal.

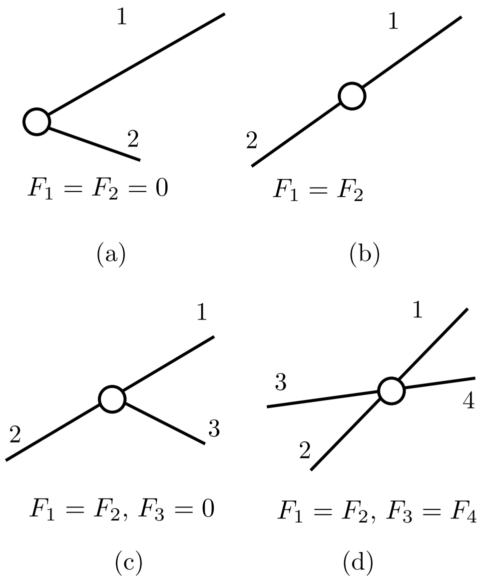

Figure 1.12 Forces in truss members.

These points are useful in predicting the forces in truss members without the actual calculations.

1.6.5 Method of Sections

Method of joints is used in determining the unknown forces on each member of a truss while method of sections is preferred in determining axial forces in only few members. In this method, the truss is cut at a section such that most of the members of unknown forces are covered. The equations of equilibrium are then applied to determine the unknown forces.

1.7 RECTILINEAR KINEMATICS

Rectilinear kinematics deals with the motion of a particle in rectilinear or straight line path. It is characterized by particle's position, velocity, and acceleration, at any given instant of time. Curvilinear motion occurs when motion of a particle follows a curved path. *Speed* is the rate of change of distance irrespective of the direction of motion of the body; *Velocity* is a vector quantity of magnitude equal to speed. *Acceleration* is the rate of change of velocity with respect to time.

Consider a particle, moving in a straight line, changes its position from $\vec{x}$ to $\vec{x}+\delta\vec{x}$ in time δt [Fig. 1.13]. Velocity and acceleration of the particle at any instant of time are defined as follows:

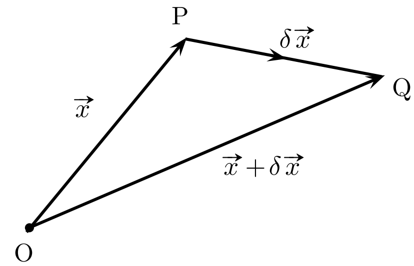

Figure 1.13 Linear displacement of a particle.

1. *Velocity* Instantaneous velocity of the particle is defined as

$$\begin{aligned}\vec{v} &= \lim_{\delta t\to 0}\frac{\delta\vec{x}}{\delta t}\\ &= \frac{d\vec{x}}{dt}\end{aligned} \tag{1.7}$$

2. *Acceleration* Instantaneous acceleration of the particle is defined as

$$\begin{aligned}\vec{a} &= \frac{d}{dt}\vec{v}\\ &= \frac{d^2\vec{x}}{dt^2}\\ &= v\frac{d\vec{x}}{dt}\end{aligned} \tag{1.8}$$

Velocity and acceleration are vector quantities. If two velocities (say $\vec{u}$ and $\vec{v}$ at angle α) are represented in magnitude and direction by two adjacent sides of a parallelogram, their resultant ($\vec{w}$) will be represented in magnitude and direction by the diagonal of the parallelogram. This is called the *law of parallelogram* [Fig. 1.14].

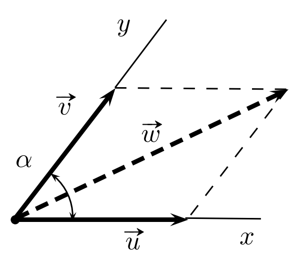

Figure 1.14 Law of parallelogram.

The magnitude of resultant velocity of u and v at angle α is given by

$$w = \sqrt{u^2+v^2+2uv\cos\alpha} \tag{1.9}$$

The law of parallelogram is based on the algebra of vectors. Therefore, it is equally valid for the forces also.

Let a particle moves at initial velocity u. In the time interval of t, it traces a distance s reaching its final velocity v. Using Eq. (1.8), the differential equation for acceleration a as

$$\frac{d^2s}{dt^2} = a$$

Integrating w.r.t. time (t),

$$\frac{ds}{dt} = at + c_1 \tag{1.10}$$

where c_1 is a constant. When $t = 0$, $ds/dt = u$, so,

$$c_1 = u$$

Therefore, Eq. (1.10) becomes

$$v = u + at \tag{1.11}$$

Integration of Eq. (1.10) gives

$$s = \frac{1}{2}at^2 + ut + c_2 \tag{1.12}$$

where c_2 is constant. When $t = 0$, $s = 0$ so $c_2 = 0$, therefore,

$$s = ut + \frac{1}{2}at^2 \tag{1.13}$$

Using Eqs. (1.11) and (1.13),

$$v^2 = u^2 + 2as \tag{1.14}$$

Equations (1.11)–(1.14) are called *equations of linear motion*.

1.8 ANGULAR MOTION

Let a particle moving in a circle travels angle $\delta\theta$ in time δt. *Angular velocity* of the particle is defined as

$$\begin{aligned}\vec{\omega} &= \lim_{\delta t \to 0} \frac{\delta \vec{\theta}}{\delta t} \\ &= \frac{d\vec{\theta}}{dt}\end{aligned} \tag{1.15}$$

Angular acceleration of the particle is defined as

$$\vec{\alpha} = \frac{d}{dt}\vec{\omega} = \frac{d^2\vec{\theta}}{dt^2} \tag{1.16}$$

Let a particle follow a circular path of radius r with constant linear speed v at angular velocity ω. So in a unit time, it runs an arc of $r\omega$, which is of the same length equal to v. Therefore,

$$\vec{v} = \vec{\omega} \times \vec{r}$$

Differentiating w.r.t. time,

$$\begin{aligned}\frac{d\vec{v}}{dt} &= r\frac{d\vec{\omega}}{dt} \\ \vec{a} &= \vec{\alpha} \times \vec{r}\end{aligned}$$

1.9 MOTION UNDER GRAVITY

Motion under gravity is described in the following subsections.

1.9.1 Universal Gravitation

Newton's law of *universal gravitation* states that any two particles or point masses attract each other along the line connecting them with a mutual force whose magnitude is directly proportional to the product of the masses and inversely proportional to the square of the distance between the particles.

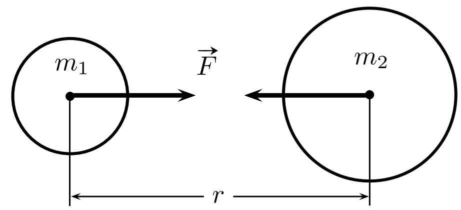

Figure 1.15 Gravitational force.

For the configuration shown in Fig. 1.15, the gravitational force F between two masses m_1 and m_2 at distance r is given by

$$F = G \times \frac{m_1 m_2}{r^2} \tag{1.17}$$

where G is the *universal constant of gravitation*, equal to 6.67×10^{11} Nm^2/kg^2.

1.9.2 Earth's Gravity

Assuming earth to be stationary and spherical body of radius R, the gravitational force of the earth (due to its mass M) acting on a body of mass m, placed at a height h above the surface of the earth, is given by

$$F = G\frac{Mm}{(R+h)^2}$$

This force is the weight of the body equal to mg. Therefore, acceleration due to gravity g is derived as

$$g = \frac{GM}{(R+h)^2} \tag{1.18}$$

The value of g is approximately 9.81 m/s^2. In engineering applications, g is usually considered as a constant and the weight force is assumed to be directly perpendicular to the earth's surface.

When a particle is projected vertically upward, there is a retardation upon it due to earth's attraction. This retardation is denoted by $-g$. When a particles falls down under gravity, it possesses an acceleration equal to g.

1.9.3 Projectile

The particle projected under gravity other than vertical is called a *projectile*. The *angle of projection* is the angle of initial velocity with horizontal plane. The path described by the particle is called *trajectory*. The *range of projectile* is the distance between the point of projection and the point where trajectory meets any horizontal plane through the projection.

Let a particle is projected upward at an angle α from horizontal at initial velocity of u [Fig. 1.16].

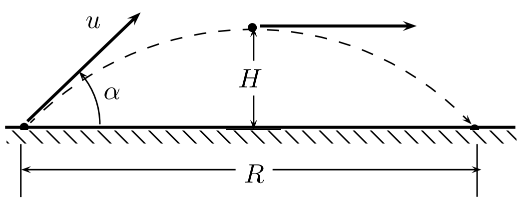

Figure 1.16 Projectile.

The following are the features of a projectile:

$$\text{Flight time } t = 2\times\frac{u\sin\alpha}{g}$$

$$\text{Maximum height } H = \frac{u^2\sin 2\alpha}{2g}$$

$$\text{Range } R = \frac{u^2\sin 2\alpha}{g}$$

The range is maximum if $\alpha = \pi/4$.

1.9.4 Vertical Projection

Consider a particle of is projected vertically upward ($\alpha = \pi/2$) at initial velocity u, and let it reaches upto height h where velocity v becomes zero. Using Eq. (1.14),

$$0^2 = u^2 + 2(-g)H$$

$$H = \frac{u^2}{2g}$$

Using Eq. (1.11), the time t taken in the reaching to the height h is determined as

$$0 = u - gt_{1/2}$$

$$t = \frac{u}{g}$$

1.10 DEPENDENT MOTION OF PARTICLES

In some types of engineering applications, motion of particles is dependent upon others. Two blocks interconnected with an inextensible spring over a pulley represent the most simple situation of dependent motion [Fig. 1.17].

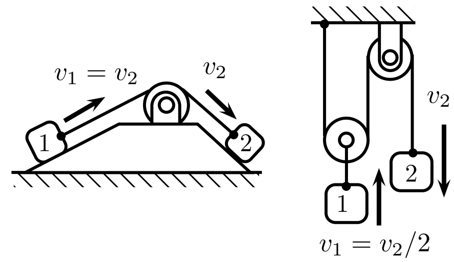

Figure 1.17 Dependent motion of two particles.

The relationship between dependent velocities can be found using constant length of the inextensible string.

1.11 NEWTON'S LAWS OF MOTION

The problems of mechanics can be solved by applying *Newton's laws of motion*[4], described as follows:

1. First Law of Motion *Newton's first law of motion* states that every object will remain at rest or in uniform motion in a straight line unless compelled to change its state by the action of an external force.

   The first law of motion is normally taken as the definition of *inertia*. If there is no net force acting on an object (if all the external forces cancel each other out) then the object will maintain a constant velocity. If that velocity is zero, then the object remains at rest. If an external force is applied, the velocity will change because of the force.

2. Second Law of Motion *Newton's second law of motion* states that if the resultant force acting on a particle is not zero, the particle will have an acceleration proportional to the magnitude of the resultant and in the direction of this resultant force. The law explains how the velocity of an

[4]Sir Isaac Newton was one of the greatest scientists and mathematicians that ever lived. He was born in England on December 25, 1643. He was born the same year that Galileo died. He was the first to formalize these laws and published them in 1986.

object changes when it is subjected to an external force. The law defines a force to be equal to change in momentum (mass times velocity) per unit time.

For an object with a constant mass m, *Newton's second law of motion* states that the force $\vec{F}$ is the product of an object's mass and its acceleration $\vec{a}$:

$$\vec{F} = m \times \vec{a}$$

For an externally applied force, the change in velocity depends on the mass of the object. A force will cause a change in velocity; and likewise, a change in velocity will generate a force. The above equation works in both ways.

3. Third Law of Motion *Newton's third law of motion* states that for every action (force) in nature there is an equal and opposite reaction. In other words, if object A exerts a force on object B, then object B also exerts an equal force on object A.

   The third law of motion can be used to explain the generation of lift by a wing and the production of thrust by a jet engine.

1.12 WORK AND ENERGY

1.12.1 Energy

The *energy* of a body is its capacity of doing work. Energy is possessed by a body, while the work is done by force on a body when it has a displacement in the direction of the force. If position vector is denoted by $\vec{r}$, then work dW, a scalar quantity, is defined as the dot product of force $\vec{F}$ and displacement vector $d\vec{r}$:

$$dW = \vec{F} \cdot d\vec{r}$$

1.12.2 Modes of Mechanical Energy

Energy can be in several forms like mechanical energy, electrical energy, heat, light, sound, pressure. The present context is of *mechanical energy*, the energy possessed by a body due to its position or motion. Hence, mechanical energy can be of two types: potential energy and kinetic energy, described as follows:

1. Potential Energy The energy which a body possesses by virtue of its position or configuration is called *potential energy*. Few examples to clarify the concept of potential energy are following:

   (a) If a body of mass m is raised through a height h above a datum[5] level, then the work done on it by the gravitational force is written as

   $$W = mgh$$

   This energy is stored in the body as potential energy. In coming down to the original position, the body is capable of doing work equal to mgh.

   (b) If a spring is twisted through an angle θ by application of a torque varying from zero in the beginning to T in the end, the work done by the average torque $T/2$ is written as

   $$W = \frac{0+T}{2} \times \theta = \frac{1}{2} T\theta$$

   This is the potential energy of the spring due to its configuration.

   (c) If a spring of stiffness k is stretched, the force F acting on it does not remain constant, but increases with displacement x undergone by the spring. At any time,

   $$F = kx$$

   Therefore, average force acting on the spring is

   $$\bar{F} = k \times \frac{0+x}{2} = k\frac{x}{2}$$

   Hence, the work done by the average force $\bar{F}$ for displacement x of the spring is written as

   $$W = k\frac{x^2}{2}$$

   This is the potential energy of the spring due to its configuration.

   Gravity force, elastic spring, and torsional spring are examples of *conservative forces*. Thus, *potential energy* is the measure of the amount of work done by a conservative force in moving a body from one position to another.

2. Kinetic Energy The energy which a body possesses by virtue of its motion is called *kinetic energy*. It is measured by the amount of work required to bring the body to rest.

   Let a body of mass m moving with velocity v be brought to rest by the application of a constant force $\vec{F}$ which causes a retardation $-\vec{a}$. If s is the distance through which the body moves in this period, the kinetic energy is given by the work done by force $\vec{F}$ on the body. Using third equation of linear motion,

   $$0^2 - v^2 = 2(-a)s$$

[5] A surface to which elevations, heights, or depths on a map or chart are related.

Therefore, kinetic energy is determined as

$$U = ma \times s$$
$$= \frac{1}{2}mv^2$$

A system of particles or body can have both forms of mechanical energy. During motion or change in the amount or direction forces, one form of energy gets converted into another form.

1.12.3 Principle of Work and Energy

The principle of work and energy states that the work done by all of the external forces and couples as a rigid body moves from position 1 to position 2 is equal to the change in the potential energy of the body:

$$\sum U_{1-2} = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

1.12.4 Principle of Conservation of Energy

The *principle of conservation of energy* states that the total amount of energy in the universe is constant; energy can neither be created nor destroyed although it can be converted into various forms.

The principle of conservation of energy can be appropriately stated as when a particle moves under the action of conservative forces, the sum of the kinetic energy and potential energy of the particle remains constant. If potential energy and kinetic energy are denoted by U and T, respectively, the principle can be stated for a system between two instances 1 and 2 as

$$T_1 + U_1 = T_2 + U_2$$

The principle of conservation of energy is generally applied to solve the problems involving forces, displacements and velocities. The principle can be applied to each element of a structure or body separately. The problems involving energy dissipation through friction and damping can be solved by considering suitable sign of the energy component of the system.

1.13 D'ALEMBERT'S PRINCIPLE

According to the *d'Alembert's principle*[6], the external forces acting on a body and the resultant *inertia* forces on it are in equilibrium. D'Alembert's principle is, indeed, a restatement of *Newton's second law of motion* but it suggests that the term $(-ma)$ can be considered as a fictitious force, often called *d'Alembert's force* or the *inertia force*. Accordingly, the net external force $\vec{F}$ actually acting on the body and the inertia force $\vec{F_i}$ together keep the body in a state of *fictitious equilibrium*:

$$\vec{F} + \vec{F_i} = 0$$

The d'Alembert's principle tends to give the solution procedure of a dynamic problem, an appearance like that of a static problem, and the above equation becomes equation of *dynamic equilibrium*.

[6]D'Alembert's principle is named after its discoverer, the French physicist and mathematician Jean le Rond d'Alembert.

1.14 IMPULSE AND MOMENTUM

1.14.1 Linear Momentum

Momentum $(\vec{p})$ is a measure of the tendency of an object to keep moving once it is set in motion. Let a particle of mass m move with a velocity $\vec{v}$ and acceleration $\vec{a}$. Using *Newton's law of motion*, the force acting on the body is given by

$$\vec{F} = m\vec{a}$$

The rate of change of momentum is

$$\frac{d\vec{p}}{dt} = \frac{d}{dt}m\vec{v}$$
$$= m\frac{d\vec{v}}{dt}$$
$$= m\vec{a}$$
$$= \vec{F}$$

This equation states that the rate of change of momentum is equal to the applied force. This statement is known as the principle of linear momentum. The law is also known as *Euler's first law*. If there are no forces applied to a system, the total momentum of the system remains constant; the law in this case is known as the *law of conservation of momentum*.

1.14.2 Angular Momentum

Angular momentum $(\vec{h})$ is the moment of momentum about an axis; it is the product of the linear momentum of the particle and the perpendicular distance from the axis of its line of action. Consider a particle of mass m moving with a velocity $\vec{v}$ and acceleration $\vec{a}$ [Fig. 1.18].

The angular momentum about an axis passing through point O at distance $\vec{r}$ is given by

$$\vec{h} = \vec{r} \times m\vec{v}$$

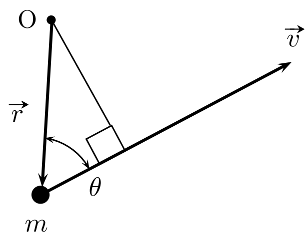

Figure 1.18 | Angular momentum.

The rate of change of angular momentum can be determined as

$$\begin{aligned}\frac{d\vec{h}}{dt} &= \frac{d}{dt}(\vec{r}\times m\vec{v})\\ &= \vec{r}\times m\frac{d\vec{v}}{dt}\\ &= \vec{r}\times\vec{F}\end{aligned}$$

This equation is known as the principle of angular momentum. It states that the resultant moment of the external forces ($\vec{F}$) acting on the system of a particle equals the rate of change of the total angular momentum of the particles. The law is also known as *Euler's second law*.

1.14.3 Impulse–Momentum Principle

If a constant force $\vec{F}$ acts for time t on a body, the product $\vec{F}\times t$ is called the *impulse* of the given force. Similarly, if a torque $\vec{T}$ acts on a body for time t, then the angular impulse is $\vec{T}\times t$.

Let a constant force F acts on a body of mass m for time t and changes its velocity from u to v under acceleration a. Then, impulse ($\vec{J}$) is given by

$$\begin{aligned}\vec{J} &= \vec{F}\times t\\ &= m\vec{a}\times t\\ &= m\times\frac{\vec{v}-\vec{u}}{t}\times t\\ &= m(\vec{v}-\vec{u})\\ &= m\vec{v}-m\vec{u}\end{aligned}$$

Therefore, *impulse–momentum principle* states that the component of resultant linear impulse along any direction is equal to change in the component of momentum in that direction.

1.15 LAW OF RESTITUTION

Impact is the collision of two particles for a very short period of time that results into relatively large impulsive forces exerted between the particles. An impact is called *central or line impact* when direction of motion of the mass centers of the two colliding particles is in a single line, otherwise, it is called *oblique impact*.

The *law of restitution* states that the velocity of separation of two moving bodies which collide with each other bears a constant ratio with their velocity of approach. The constant of proportionality is called the *coefficient of restitution*, denoted by e. This property, first discovered by Newton, is known as the *Newtons law of restitution*.

Consider two particles moving with initial velocities u_1 and u_2 towards each other. These particles collide on center-line (center impact), and after impact, their respective velocities become v_1 and v_2 [Fig. 1.19].

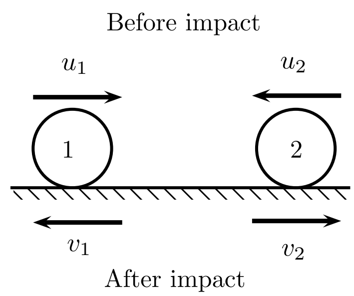

Figure 1.19 | Coefficient of restitution.

The coefficient of restitution (e) is expressed as the ratio of relative velocities of the particles' separation just after impact ($v_2 - v_1$) to the relative velocity of the particles' approach just before impact ($u_1 - u_2$):

$$e = \frac{v_2 - v_1}{u_2 - u_1}$$

Experiments show that e varies appreciably with impact velocity as well as with the size and shape of the colliding particles, ranging from 0 to 1. The value of the coefficient of restitution has got physical meaning. Based on the limiting values of e, the collision can be classified into two types:

1. Elastic Collision A *perfectly elastic collision* occurs without loss of kinetic energy of the particles. Thus, for elastic collisions, $e = 1$.
2. Inelastic Collision A *inelastic collision* or *plastic collision* is one in which part of the kinetic energy is changed to some other form of energy in the collision.

   Momentum is conserved in inelastic collisions, however, the kinetic energy in the the collision is converted into other forms of energy. For inelastic collisions, $e = 0$.

The principle of work and energy cannot be used for the analysis of impact problems because it is impossible to know the variation in the internal forces of deformation and restitution during the collision. The energy loss can be calculated as the change in kinetic energy of the particles.

1.16 PRINCIPLE OF VIRTUAL WORK

When the point of application of a force is imagined to be displaced through a differential distance in the direction of the force, the imaginary work done by the force is called *virtual work*.

The *principle of virtual work* states that the work done on a rigid body or a system of rigid bodies in equilibrium is zero for any virtual displacement compatible with the constraints on the system. Conversely, if the virtual work for all such displacements is zero, then the body is in equilibrium.

Virtual displacement is an imaginary infinitesimal displacement. A differential virtual displacement is denoted by δ to distinguish it from a differential displacement generally denoted by d.

The method of virtual work is explained by two examples:

1. Consider a rod AB which can rotate about a fulcrum O [Fig. 1.20]. A vertical load F_1 is applied at end A. It is required to calculate the a vertical force F_2 to be applied at end B to keep the rod in current position. In virtual work method, the body

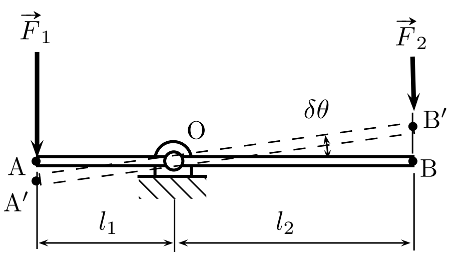

Figure 1.20 Virtual displacement.

is assumed to be virtually displaced. For present case, let the rod undergo a virtual rotation through angle $\delta\theta$ about the fulcrum O to assume the new position A′B′. The total virtual work during this rotation is given by

$$\delta W = F_1 \times l_1 \delta\theta - F_2 \times l_2 \delta\theta$$

According to the principle of virtual work, total virtual work must be zero, therefore,

$$F_1 \times l_1 \delta\theta - F_2 \times l_2 \delta\theta = 0$$
$$F_2 = F_1 \frac{l_1}{l_2}$$

2. Consider a lazy tong mechanism [Fig. 1.21]. The joint A has a pin which is free to slide inside the vertical groove provided in the frame. The joint E has a torsional spring to keep the mechanism in equilibrium under the external force $\vec{F}$ applied at the hinge joint E.

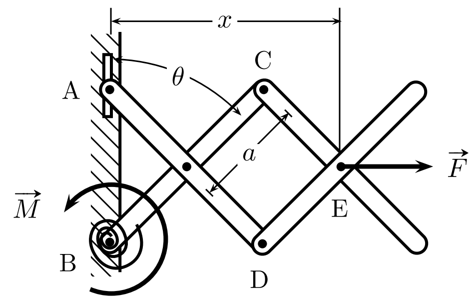

Figure 1.21 Lazy tong mechanism.

The magnitude of the moment M required to keep the mechanism in equilibrium can be determined using the method of virtual work. The horizontal distance x of joint E from the AB is

$$x = 3a\cos\theta$$

The virtual rotation of link BC for virtual displacement of δx of the joint E is given by

$$\delta x = -3a\sin\theta \times \delta\theta$$

The reactions at the joints A and B will not cause any work, the total virtual work done by moment M and external force F must be zero:

$$M\delta\theta + F\delta x = 0$$
$$M\delta\theta - F \times 3a\sin\theta \times \delta\theta = 0$$
$$M = 3Fa\sin\theta$$

In applying the method of virtual work, it is necessary to only calculate the displacements of the points of application of the forces, and hence a problem of equilibrium is converted into one of geometry, which is usually easier to solve. Also, forces whose points of application are not displaced, or the displacement is perpendicular to the force, need not be considered. The superiority of the method of virtual work is that the method eliminates all unknown reactions.

There is no added advantage of applying the principle of virtual work in equilibrium problems. Each application of the virtual work equation, leaves an equation that could have been directly obtained by simply applying the equation of equilibrium.