CHAPTER 4

VIBRATIONS

Vibrations are the oscillations of a structural system about the equilibrium position. In general sense, these are *periodic motions*, repeating in a certain interval of time. All the structural system possessing mass and elasticity are capable of vibrations to some extent. If uncontrolled, vibrations can lead catastrophic situations and unusual consequences. Vibrations are induced by unbalanced forces and can also be induced for benefits. Therefore, the design of engineering systems requires consideration of the vibrational factors. Vibration isolators are used to protect structures from excessive forces developed in the operation of rotating machine. The theory of vibrations is concerned with the study of oscillatory motions of bodies and the forces associated with them.

4.1 FUNDAMENTALS

This section describes the basic concepts and analysis of mechanical vibrations.

4.1.1 Basic Phenomenon

All bodies having mass and elasticity are capable of vibrations. Mass is an inherent property of the body. Elasticity of the material permits relative motion among its parts. When body particles are displaced from the equilibrium position by the application of external force, the internal forces of the body, in the form of stresses and inertia, try to bring the body to its original equilibrium position. The swinging of pendulum and the motion of plucked spring are typical examples.

Vibration of a system involves transfer of potential energy to kinetic energy, and kinetic energy to potential energy. In a damped system, some energy is dissipated in each cycle of vibration which must be replaced by an external source, if state of steady vibration is to be maintained.

4.1.2 Harmonic Motion

Harmonic motion is the simplest form of vibrations which is represented in terms of *amplitude* x_0, time t and *frequency* ω, and phase angle ϕ in trigonometric function [Fig. 4.1]:

$$x = x_0 \sin(\omega t + \phi) \tag{4.1}$$

A harmonic motion having amplitude x_0 and rotating at constant angular velocity ω can be represented in exponential form or as a complex quantity

$$\begin{aligned} z &= x_0 e^{j\omega t} \\ &= x_0 \cos\omega t + j x_0 \sin\omega t \end{aligned} \tag{4.2}$$

where z is referred as *complex sinusoid*.

The associated terms of sinusoidal representation are defined as follows:

1. *Amplitude* *Amplitude* (x_0) is the maximum displacement of a particle under the harmonic motion from equilibrium position. This peak value indicates the maximum strain that the vibrating part is undergoing. The average value of displacement can be found as

$$\bar{x} = \lim_{T\to\infty} \frac{1}{T} \int_0^T x(t)\, dt$$

The square of the displacement is associated with the energy of the vibration for which the mean square value is a measure:

$$\bar{x}^2 = \lim_{T\to\infty} \frac{1}{T} \int_0^T x^2(t)\, dt$$

2. *Period* *Period* is the time required to execute one cycle of the harmonic motion.

3. *Frequency* *Frequency* (f) of a harmonic motion is the number of cycles executed in unit time. It is the inverse of time period (T):

$$f = \frac{1}{T}$$

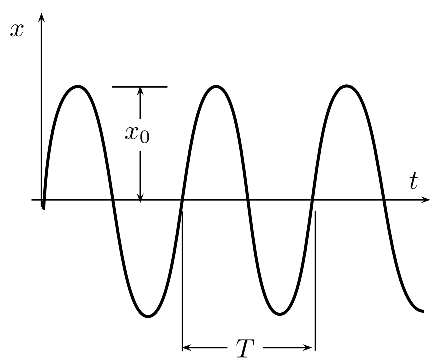


Figure 4.1 | Harmonic motion.

The unit of frequency, cycle per second, is designated as one hertz (Hz).

As the system executes one cycle, the argument of the trigonometric function runs through 2π radians. Thus,

$$1 \text{ cycle} = 2\pi \text{ radians}$$

Therefore, the periodicity of motion is also expressed in terms of *circular frequency*, measured in rad/s:

$$f = \frac{\omega}{2\pi} \text{ cycles/s}$$
$$= \omega \text{ rad/s}$$

Frequency is also expressed in terms of revolution per minute (rpm):

$$f = \frac{60\omega}{2\pi} \text{ rpm}$$

4. *Phase Angle* *Phase angle* (ϕ) represents the lead or lag between the response and a purely sinusoidal response. If $\phi > 0$, the response is said to *lag* a pure sinusoid, and if $\phi < 0$, the response is said to *lead* the sinusoid.

4.1.3 Work Done per Cycle

Let a vibrating force $F = F_0 \sin \omega t$ act on a particle and causes displacement $x = x_0 \sin(\omega t - \phi)$. In one cycle of the harmonic motion, the system executes 2π radian. Therefore, *work done per cycle* is determined[1] as

$$\begin{aligned} W &= \int_0^{2\pi/\omega} F \frac{dx}{dt} dt \\ &= \int_0^{2\pi/\omega} F_0 \sin \omega t \times \frac{d}{dt} \{x_o \sin(\omega t - \phi)\}\, dt \\ &= \int_0^{2\pi/\omega} F_0 \sin \omega t \times x_0 \omega \cos(\omega t - \phi)\, dt \\ &= F_0 x_0 \omega \int_0^{2\pi/\omega} \sin \omega t \cos(\omega t - \phi)\, dt \\ &= \frac{F_0 x_0 \omega}{2} \int_0^{2\pi/\omega} (\sin 2\omega t + \sin \phi)\, dt \\ &= \frac{F_0 x_0 \omega}{2} \left[-\frac{\cos 2\omega t}{2\omega} + \sin \phi \times t \right]_0^{2\pi/\omega} \\ &= \frac{F_0 x_0 \omega}{2} \left[-(0-0) + \left(\sin \phi \times \frac{2\pi}{\omega} - 0 \right) \right] \end{aligned}$$

This finally results

$$W = \pi F_0 x_0 \sin \phi \tag{4.3}$$

This indicates that the work done per cycle in by a harmonic force depends upon the phase difference between the force and displacement. Eq. (4.3) can be examined for two extreme conditions:

[1] Using

$$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$$

$$\int \sin \theta d\theta = -\cos \theta$$

1. When force and displacement functions are in same phase ($\phi = 0$):

$$W = 0$$

2. When force and displacement functions are orthogonal ($\phi = \pi/2$):

$$W = \pi F_o x_o$$

4.1.4 Superposing Waves

Figure 4.2 depicts two *sinusoidal waves* in a polar or vector diagram:

$$x_1 = x_{01} \sin \omega_1 t$$
$$x_2 = x_{02} \sin \omega_2 t$$

The relative phase angle $(\omega_1 - \omega_2)\, t$ is the angle between these vectors.

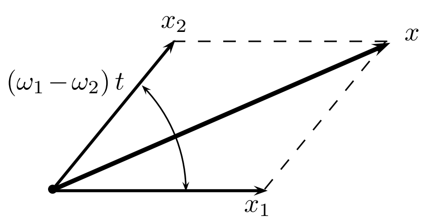


Figure 4.2 | Superposing waves.

If these waves coincide on a common medium in the same direction, the resulting wave of superposition is given by their vector sum:

$$\begin{aligned} x &= x_1 + x_2 \\ &= x_{01} \sin \omega_1 t + x_{02} \sin \omega_2 t \\ &= x_0 \sin (\omega_2 - \omega_1)\, t \end{aligned}$$

where

$$x_0 = \sqrt{x_{01}{}^2 + x_{02}{}^2 + 2 x_{01} x_{02} \cos (\omega_2 - \omega_1)\, t}$$

4.1.5 Classification of Vibrations

Some of the important attributes of classifications are as follows.

4.1.5.1 Degrees of Freedom The number of degrees of freedom of a mechanical system is the number of kinematically independent coordinates necessary to completely describe the motion of each element of the system. Based on this, the vibration systems can be classified into the following:

1. Discrete Systems A vibration system having a finite number of degrees of freedom is called *discrete system*.
2. Single Degree of Freedom A system having only one degree of freedom is called a *single degree of freedom* (SDOF) system.
3. Multiple Degree of Freedom A system with two or more degree of freedom is called a *multiple degree of freedom* (MDOF) system.
4. Continuous System A system with an infinite number of degrees of freedom is called a *continuous system*.

The number of degrees of freedom indicates the number of differential equations or variables required to define a system. Therefore, complexity in predicting the behavior of a system increases with increase in the number of degrees of freedom.

4.1.5.2 Characteristics Linearity and non-linearity of a mechanical system directly affect the difficulty in predicting the behavior of system, discussed as follows:

1. Linear System If all the basic components of a vibratory system: the spring, the mass, and the damper, behave linearly, the resulting vibration is governed by linear differential equations. Therefore, such a vibration is known as *linear vibration*. The principle of superposition is valid for linear systems only.
2. Non-linear System A system is *non-linear* if its motion is governed by non-linear differential equations. This can be caused by the non-linear behavior of one or more components of the system. The principle of superposition is invalid for non-linear vibrations.

Mathematical techniques and methods are well developed for analysis of linear systems while those for non-linear systems are still under development.

4.1.5.3 External Inputs The vibrations can be free or forced, described as follows.

1. Free Vibrations In a *free vibration*, the system oscillates under the action of inherent inertia and elastic forces of the system, initiated by a small disturbance; vibrations occur in the absence of external forces. The oscillation of a simple pendulum is a typical example of free vibrations.

   If a system is left after an initial disturbance to vibrate on its own, the frequency with which it oscillates naturally, without external forces, is known as the *natural frequency* of the system. A vibratory system having n degrees of freedom will have n distinct natural frequencies of vibration.

2. Forced Vibrations In contrast to free vibrations, *forced vibrations* take place under the excitation of external forces. The oscillations in machines, such as engines, are forced vibrations.

   If the frequency of the external force coincides with the natural frequencies of the system, a condition known as *resonance* occurs; the system undergoes dangerously large oscillations. Failure of structures, such as building, bridges, turbines, and airplane wings, is generally associated with the occurrence of resonance.

   If the external force is periodic, the vibrations are *harmonic*. If the external input is aperiodic, vibrations are said to be *transient*. If the excitation force is known at all times, the excitation is said to be *deterministic*. If the excitation force is *stochastic* (unknown, unpredictable) the excitation is said to be *random* or *non-deterministic*. Examples of random excitations are wind velocity, road roughness, and ground motion during earthquakes. In these cases, a large collection of records of the excitation can exhibit some statistical regularity. It is possible to estimate averages, such as the mean and mean square values of the excitation.

4.1.5.4 Energy Dissipation A vibration system can have elements that dissipate energy. In this respect, the systems are classified as follows:

1. Undamped Vibrations Vibrations without energy dissipation are called *undamped vibrations*.

2. Damped Vibrations If an energy dissipating element is present in the system, the vibrations are called *damped vibrations*.

In many physical systems, the amount of damping is so small that it can be disregarded for most of the engineering purposes. However, consideration of damping becomes extremely importance in analyzing systems near resonance.

4.1.6 Elements of Vibration Systems

A mechanical system consists of three basic elements: inertia, stiffness, and damping. *Inertia* components store kinetic energy, *stiffness* components store potential energy, and *damping* components dissipate the energy of the system. In addition to these, external forces provide energy to the system. These elements are described under the following subsections.

4.1.6.1 Spring Elements Springs act as reservoir of potential energy, the energy by virtue of displacement or deflection, but they don't require motion (velocity) to do so. A helical-coil spring serves as the model for all linear structural components, such as bars undergoing longitudinal motion, shafts under rotational motion, and beams under transverse vibrations; all store potential energy and can be modeled as springs. The characteristics of a stiffness component are described as follows:

1. Stiffness A spring is a flexible mechanical link between two particles in a mechanical system. In reality, a spring itself is a continuous system. However, the inertia of the spring is usually small compared to other elements in the mechanical systems, and is neglected. Under this assumption, the force applied to each end of the spring is the same.

   The length of the spring when it is not subjected to external force is called *unstretched length*. Since the spring is made of a flexible material, the force F that must be applied to the spring to change its length by x should be continuous function of x. A linear spring obeys a force-displacement law in following format:

$$F = kx$$

   where k is called the *spring stiffness* or *spring constant* which has dimensions of force per unit length.

   Using the *constitutive equation*, the stiffness of a structural member can be appropriately defined in the following form:

$$k = \left(\frac{dF}{dx}\right)_{x=0}$$

   For example, consider a cantilever beam of length l with an end mass m. For simplicity, mass of the beam is assumed negligible. From the subject of strength of material, the static deflection of the beam at the free end is given by

$$\delta = \frac{Wl^3}{3EI}$$

   where, $W(= mg)$ is the weight of the mass, E is Young's modulus of beam material, and I is the moment of inertia of the cross-section of the beam. Therefore, the spring constant for the system is

$$\begin{aligned} k &= \frac{W}{\delta} \\ &= \frac{3EI}{l^3} \end{aligned}$$

   The modeling of stiffness components in the form of combination of springs (in series or parallel) is convenient by such components by a single spring of an equivalent stiffness k_e such that the

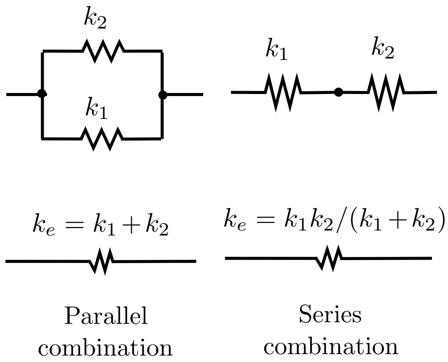


Figure 4.3 | Combination of springs elements.

system undergoes same displacement for a given force [Fig. 4.3].

2. <u>*Potential Energy*</u> The work done by a force is calculated as force multiplied by distance. Figure 4.4 shows the work done by the spring force as its point of application moves from a position x_1 to x_2. This work is stored as potential energy (U) in the spring, given by

$$U_{1\to 2} = \int_{x_1}^{x_2} (-kx)\, dx$$
$$= k\frac{{x_1}^2}{2} - k\frac{{x_2}^2}{2}$$

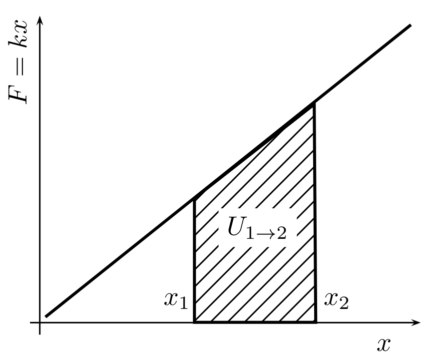


Figure 4.4 | Potential energy in springs.

Since the work depends upon the initial and final positions of the point of application of the spring force and not on the path of the system, the spring force is *conservative* in nature. Therefore, the potential energy function[2] can be defined for spring as

$$U(x) = \frac{1}{2}kx^2$$

where x is the change in the length of the spring from its original length.

4.1.6.2 Inertia Elements The inertia element of a mechanical is assumed to be a rigid body which acts as a reservoir of kinetic energy, the energy by virtue of the velocity of the body. Using the *Newton's second law of motion*, the product of the mass and its acceleration is equal to the force applied to the mass. The work is equal to the force multiplied by the displacement in the direction of the force. The work done on a mass is stored in the form of the kinetic energy (T).

The mass of a body acts as inertia force against the linear motion. For angular motions, distribution of mass in the body affects the inertia against the rotation. For this, *moment of inertia* (I) is used as measure of inertia in angular motions, defined as

$$I = \int_0^r r^2 dm$$

where r is the distance of center of infinitesimal mass dm from the axis of rotation of the body [Fig. 4.5].

According to *d'Alembert's principle*, while dealing with dynamics, inertia force or torques ($m\ddot{x}$ and $I\ddot{\theta}$) should be taken into account.

4.1.6.3 Damping Elements In many practical systems, the vibrational energy is gradually converted to heat or sound, which results into a gradual reduction in the energy and the amplitude of the vibrations. The mechanism of gradual conversion of the vibrational energy is known as *damping*. A damper is assumed to have neither mass nor elasticity. Damping forces exist only if there is a relative velocity between two ends of the damper.

There are mainly four types of damping mechanisms used in mechanical systems: viscous damping, Coulomb damping, structural damping, and slip damping. These are explained as follows:

1. <u>*Viscous Damping*</u> *Viscosity* is the property of a fluid by virtue of which it offers resistance to the motion of one layer over the adjacent one. Some amount of energy is dissipated in overcoming

[2]Similarly, a *torsional spring* is a link in a mechanical system where application of torque leads to an angular displacement between the ends of the torsional spring. A linear torsional spring has a relationship between an applied moment (M) and the angular displacement (θ) as

$$M = k_t\theta$$

where k_t is the torsional stiffness which has dimensions of force times length. Therefore, the potential energy function for a torsional spring is

$$U(\theta) = \frac{1}{2}k_t\theta^2$$

which is similar to that of linear spring.

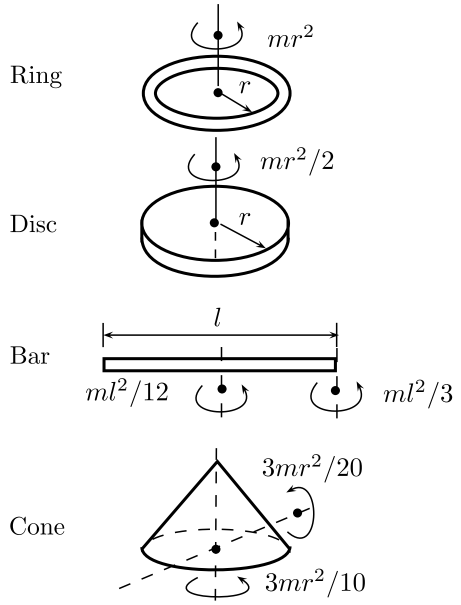


Figure 4.5 Moment of inertia.

this viscous resistance. Therefore, when a system is allowed to vibrate in a viscous medium, the resulting damping is called as *viscous damping*[3].

Consider two plates of equal area A separated by a fluid film of coefficient of viscosity μ and thickness t. The upper plate is allowed to move parallel to the fixed plate with a velocity $\dot{x}$ [Fig. 4.6].

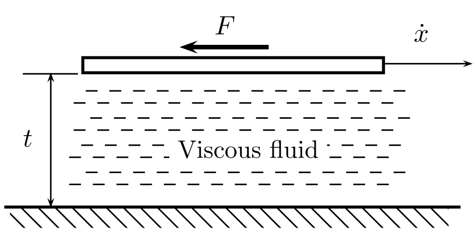


Figure 4.6 Viscous damping.

Using *Newton's law of viscosity*, the net viscous force F required to maintain this motion is expressed as

$$\begin{aligned} F &= \frac{\mu A}{t}\dot{x} \\ &= c\dot{x} \end{aligned}$$

where c is called the *viscous damping coefficient*, used as the measure of viscous damping.

The equivalent damping coefficient for a combination of viscous dampers can be determined as in case of springs [Fig. 4.7].

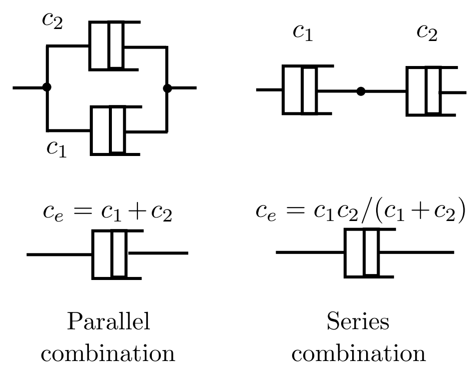


Figure 4.7 Combination of viscous dampers.

The rate of energy dissipated per cycle is determined as

$$\begin{aligned} E &= \oint F \cdot dx \\ &= \oint c\dot{x}dx \\ &= \int_0^{2\pi/\omega} c\dot{x}^2 dt \end{aligned} \tag{4.4}$$

The primary objective of damping in oscillatory systems is to limit the amplitude of the vibration at resonance. For a simple harmonic motion:

$$\begin{aligned} x &= x_0 \sin \omega t \\ \dot{x} &= \omega x_0 \cos \omega t \end{aligned}$$

Therefore, the amplitude at resonance can be represented as

$$\begin{aligned} F &= c\dot{x} \\ F &= c x_0 \omega \\ x_0 &= \frac{F}{c\omega} \end{aligned}$$

Using Eq. (4.4), the energy dissipated per cycle is

$$\begin{aligned} E &= \int_0^{2\pi/\omega} c\omega^2 x_0^2 \left(\frac{1+\cos 2\omega t}{2}\right) dt \\ &= \pi c \omega x_0^2 \end{aligned}$$

[3]If a non-ferrous conducting object is moved in a direction perpendicular to the lines of magnetic flux which is produced by a permanent magnet, then as the object moves, eddy current, proportional to the velocity, is induced in the object. This eddy current sets up a magnetic field so as to oppose the original magnetic field that has induced it. This provides a resistance to the motion of the object in the magnetic field. For analysis purposes, this is also considered mechanical damping of viscous type.

Thus, energy dissipation per cycle under viscous damping is proportional to the square of the amplitude of motion, therefore, the hysteresis curve is an ellipse [Fig. 4.8]. For non-linear damping, the energy is not a quadratic function of amplitude, therefore, the curve is no longer an ellipse.

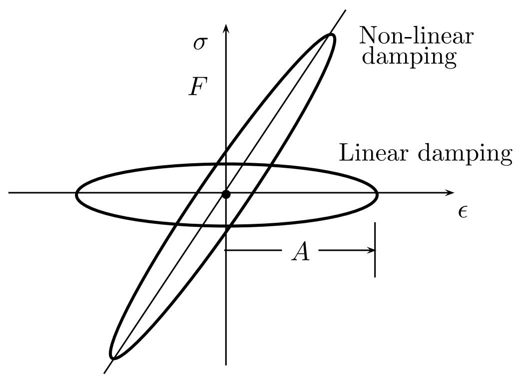


Figure 4.8 | Hysteresis in viscous damping.

The damping properties of a material can also be presented in alternatives forms:

(a) *Specific Damping Capacity* - *Specific damping capacity* is defined as the energy loss per cycle divided by the peak potential energy:

$$\beta = \frac{E}{U}$$

(b) *Loss Coefficient* - *Loss coefficient* is defined as the ratio of damping energy loss per radian divided by the peak potential or strain energy:

$$\eta = \frac{E}{2\pi U}$$

For non-viscous damping, no such simple expression exists. However, an equivalent damping coefficient c_e can be determined by equating the energy dissipated by the viscous damping to that of non-viscous damping, assuming harmonic motion.

2. Coulomb Damping When a body is allowed to slide over the other, the surfaces offer frictional resistance to the relative motion. Some amount of energy is always dissipated in overcoming the friction. The damping induced by friction is called *Coulomb damping* or *dry friction damping*.

The general expression for coulomb damping is

$$F = \mu R_n$$

where μ is the coefficient of friction, and R_n is the normal reaction.

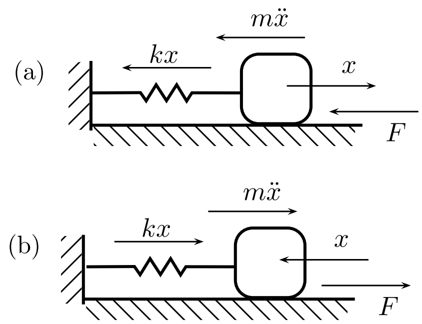


Figure 4.9 | Coulomb damping.

Consider a spring-mass system subjected to Coulomb damping [Fig. 4.9].

The reduction in the amplitude of the vibration is examined in two directions separately:

(a) Rightward Movement For the rightward movement of mass m connected with spring of stiffness k, the equilibrium equation is

$$m\ddot{x} + kx + F = 0$$

By applying the boundary conditions: at $t = 0$, $x = x_0$ and $\dot{x} = 0$,

$$x = \left(x_0 - \frac{F}{k}\right) \cos \sqrt{\frac{k}{m}} t + \frac{F}{k}$$

The equation holds good for half cycle. When $t = \pi/\omega$ ($\cos \pi = -1$), the half cycle gets completed during which the displacement is obtained from above equation as

$$x_{1/2} = -\left(x_0 - \frac{2F}{k}\right)$$

Thus, the initial amplitude x_0 is reduced by $2F/k$ in half cycle. The natural frequency of oscillation of the system is $\omega_n = \sqrt{k/m}$.

(b) Leftward Movement The dynamic equation for leftward movement of the mass, for reversed sign convention of x' $(= -x)$, is

$$m\ddot{x} + kx + F = 0$$

Therefore, the amplitude again reduces by $2F/k$ in the half cycle. The natural frequency of oscillation of the system remains constant at $\omega_n = \sqrt{k/m}$.

The decay in amplitude per cycle in Coulomb damping is found as

$$\Delta = \frac{4F}{k}$$

This is a constant quantity for constant friction force F and stiffness k. The motion will cease, however, when the amplitude becomes less than the elongation of spring at which the spring force is insufficient to overcome the static friction force.

3. *Structural Damping* *Structural damping* is offered by the elastic properties of the structure itself. This type of damping is due to the internal friction of the molecules of elastic materials. When a material is subjected to cyclic reversal of loading, a hysteresis loop appears on the stress–strain diagram, indicating that more work is required for straining the material than what is recovered during the return of the cycle [Fig. 4.10]. The

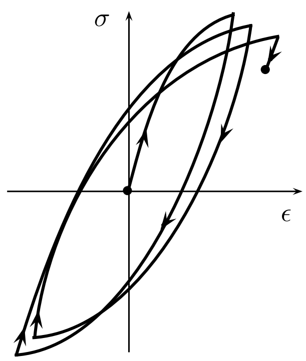


Figure 4.10 Hysteresis loop in cyclic loading.

difference of the work is measured by the area of the hysteresis loop as the energy dissipated per unit volume per cycle. Therefore, structural damping is also called *hysteresis damping*. The magnitude of this type of damping is very small as compared to that of other modes of damping.

Experiments indicate that the energy dissipated per cycle is proportional to the stiffness of the material (k) and square of the amplitude (x_0), but independent of frequency of the vibration:

$$E = \alpha x_0^2$$

where k is a constant (having units of force per unit displacement) representing the influence of shape, size, and stiffness of the structure. The equivalent damping coefficient can be determined as

$$\pi c_e \omega x_0^2 = \alpha x_0^2$$
$$c_e = \frac{\alpha}{\pi\omega}$$

The structural damping *loss coefficient* is

$$\eta = \frac{E}{2\pi U} = \frac{\alpha x_0^2}{2\pi\left(k x_0^2/2\right)} = \frac{\alpha}{\pi k}$$

where k is the stiffness of the structure.

4. *Slip Damping* Damping is also caused by the friction between the internal planes of a structure, that slip or slide as the deformation takes place. Microscopic slip occurs on the interfaces of machine elements which causes dissipation of vibration energy. This results into damping of vibrations which is called *slip damping*.

4.2 UNDAMPED FREE VIBRATION

Resonance is the situation when natural frequency of vibration coincides with that of excitation in a given machine. Therefore, determination of natural frequency and amplitude of vibrations of a machine element is essential in designing process.

The following are the three basic methods employed for vibrational analysis:

1. Equilibrium method
2. Energy method
3. Rayleigh's energy method

These are described in the following sections along with the examples of undamped free vibrations of single degree of freedom.

4.2.1 Equilibrium Method

The equilibrium method considers the equilibrium of external and internal forces in the system. It is also known as *Newton's method* because it employs the *Newton's second law of motion*. The law states that the rate of change of momentum of a mass is equal to the force acting on it.

$$F_i = m\ddot{x}$$

This force F is the the inertia force, as explained by the *d'Alembert's principle*. The principle states that a body, which is not in static equilibrium by virtue of some displacement, can be brought to static equilibrium by introducing on it an inertia force which is equal to the mass times the acceleration of the body and acts through the center of gravity of the body but in opposite direction to the acceleration. Therefore, in static equilibrium, the vector sum of the resultant external force (F) acting on a body and the inertia force (F_i) is equal to zero:

$$F + F_i = 0$$

Equivalent or extended form of Newton's method is the *principle of virtual work*. For example, when mass of a spring-mass system is given a virtual displacement δx, the total virtual work done by all the forces is set equal to zero to obtain

$$-k\ddot{x}\delta x - kx\delta x = 0$$
$$m\ddot{x} + kx = 0$$

The following two examples explain the general procedure of the equilibrium method:

1. Spring-Mass System Consider a spring mass system constrained to move in a rectilinear manner along the axis of the spring. Spring of constant stiffness k is fixed at one end and carries a mass m at its free end [Fig. 4.11].

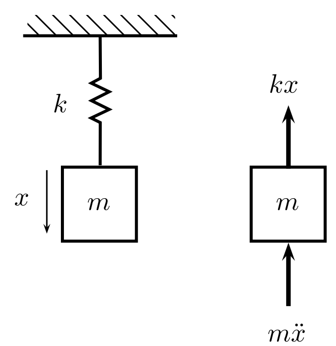


Figure 4.11 Spring-mass system.

The body is displaced x distance from its equilibrium position vertically downwards. This equilibrium position is called *static equilibrium*. The spring force kx and the inertia force $m\ddot{x}$ both act in upward direction. For equilibrium,

$$m\ddot{x} + kx = 0 \tag{4.5}$$

This is the differential equation of motion of the spring-mass system, which can be solved for x. Therefore, the natural frequency of the system is

$$\omega_n = \sqrt{\frac{k}{m}} \tag{4.6}$$

2. Simple Pendulum Consider a pendulum system consisting of a hanging body of mass m attached to a massless string of length l [Fig. 4.12].

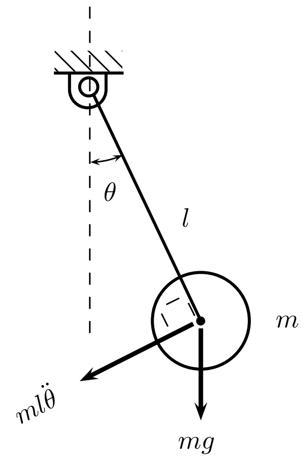


Figure 4.12 Pendulum.

If mass is displaced at an small angle θ, then the equation of equilibrium of forces acting on the mass is found as

$$ml\ddot{\theta} + mg\theta = 0$$

This equation can be solved for θ. The natural frequency of the pendulum is found as

$$\omega = \sqrt{\frac{g}{l}}$$

4.2.2 Energy Method

Kinetic energy and potential energy are the two forms of microscopic energy of a system, which can be related to motion and the influence of some external effects:

1. Kinetic Energy The energy that a system possesses by virtue of its motion relative to some reference frame is called *kinetic energy* (T). When all the parts of a system, having mass m, move with the same velocity $\dot{x}$ with respect to some fixed reference frame, the kinetic energy is expressed as

$$T = m\frac{\dot{x}^2}{2}$$

2. Potential Energy The energy that a system possesses by virtue of its elevation in a gravitational field is called *potential energy* (U). The gravitational field can be gravity, magnetism, electricity, or surface tension. When all parts of a system, having mass m, are at elevation x relative to center of a potential field, say gravity g, the potential energy of the system is expressed as

$$\mathrm{U} = mgx$$

In a vibratory system, the energy is partly potential and partly kinetic. *Energy method* considers the system as conservative; no energy is lost due to friction or energy dissipating non-elastic members. Thus, the sum of the kinetic energy and potential energy is constant:

$$T+U = \text{constant}$$

$$\frac{d}{dt}(T+U) = 0 \tag{4.7}$$

Differentiation of the above equation w.r.t. time (t) gives the differential equation of the equilibrium of the system. Following two examples explain the general procedure of energy method:

1. Spring-Mass System   Consider a spring–mass system constrained to move in a rectilinear manner along the axis of the spring. Spring of constant stiffness k is fixed at one end and carries a mass m at its free end [Fig. 4.11]. The potential energy and kinetic energy at any instant of time will be given by

$$U = \frac{1}{2}kx^2$$
$$T = \frac{1}{2}m\dot{x}^2$$

Using Eq. (4.7),

$$\frac{d}{dt}\left\{\frac{1}{2}m\dot{x}^2+\frac{1}{2}kx^2\right\} = 0$$
$$m\ddot{x}+kx = 0 \tag{4.8}$$

This equation of motion for the spring-mass system is the same as Eq. (4.5).

2. Cylinder Rolling on Cylindrical Surface   Consider a solid cylinder of radius r rolling without slipping on a cylindrical surface of radius R [Fig. 4.13].

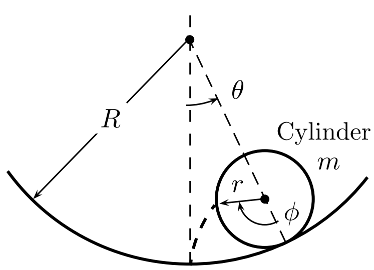


Figure 4.13 Cylinder on cylindrical surface.

Let the cylinder roll (without slipping) by angle ϕ during which it traces an angle θ at the center of cylindrical surface. Therefore,

$$r\phi = R\theta$$
$$r\dot{\phi} = R\dot{\theta}$$

During rolling, both the translation and rotation of the cylinder take place by the following velocities:

(a) Translational velocity

$$\dot{x} = (R-r)\,\dot{\theta}$$

(b) Rotational velocity

$$\omega = \dot{\phi}-\dot{\theta} = \left(\frac{R}{r}-1\right)\dot{\theta}$$

If m is the mass of cylinder, kinetic energy (T) and potential energy (U) of the system at any angle θ are written as

$$T = \frac{1}{2}m\dot{x}^2+\frac{1}{2}\left(mr^2\right)\omega^2 = \frac{3}{4}(R-r)^2\,\dot{\theta}^2$$
$$U = mg\,(R-r)\,(1-\cos\theta)$$

For small values of θ: $\sin\theta \approx \theta$, therefore, using Eq. (4.7),

$$\ddot{\theta}+\frac{2g}{3\,(R-r)}\theta = 0$$

Natural frequency of the oscillations is found as

$$\omega_n = \sqrt{\frac{2g}{3\,(R-r)}}$$

4.2.3 Rayleigh's Energy Method

The principle of conservation of energy for an undamped system [Eq. (4.7)] is restated as

$$T+U = \text{constant}$$

This can have an alternative form that the maximum kinetic energy at the mean position will be equal to the maximum potential energy at the extreme position:

$$T_{max} = U_{max} \tag{4.9}$$

The application of this equation is known as Rayleigh's energy method, which directly gives the natural frequency of the system.

The method has an alternative form. If motion of various masses of a system can be expressed in terms of a single displacement x of some specific point; the system is simply one of a single degree of freedom, the kinetic energy of the system can be written as

$$T = \frac{1}{2}m_e\dot{x}^2$$

where m_e is the *effective* or *equivalent lumped mass* at the specified point. For the equivalent stiffness k_e of the system at the specified point, the natural frequency can be written as

$$\omega_n = \sqrt{\frac{k_e}{m_e}} \tag{4.10}$$

This is evident in the following examples:

1. *Spring–Mass System* For the spring-mass system [Fig. 4.11], the displacement is represented as

$$x = x_0 \sin \omega_n t$$

Differential of above equation w.r.t. time (t) gives expression of velocity as

$$\dot{x} = \omega_n x_0 \cos \omega_n t$$

Maximum velocity at mean position is $\omega_n a$, therefore, maximum kinetic energy at mean position is $m(\omega_n x_0)^2/2$. Similarly, the maximum potential energy at the extreme position is $kx_0^2/2$. Thus,

$$\frac{1}{2} m (\omega_n x_0)^2 = \frac{1}{2} k x_0^2$$

Therefore, the natural frequency of the system is

$$\omega_n = \sqrt{\frac{k}{m}}$$

This is the same as obtained in Eq. (4.6).

2. *Effect of Mass of Spring* Consider the above example when mass of the spring (m_s) is not ignorable. Length of spring is l. The velocity of any spring element at distance y from the base can be assumed in a linear fashion as

$$\dot{y} = \frac{y}{l}\dot{x}$$

Therefore, the kinetic energy of the spring–mass system can be written as

$$T_s = \frac{1}{2} m\dot{x}^2 + \frac{1}{2}\int_0^l \left(\dot{x}\frac{y}{l}\right)^2 \frac{m_s}{l} dy$$
$$= \frac{1}{2}\left(m + \frac{m_s}{3}\right)\dot{x}^2$$

The equivalent mass of the system is found as

$$m_e = m + \frac{m_s}{3}$$

Taking $k_e = k$, natural frequency of the vibrations is found using Eq. (4.10),

$$\omega_n = \sqrt{\frac{k}{m + m_s/3}}$$

4.3 FREE DAMPED VIBRATION

In the absence of energy dissipation, a free vibration can persist forever. Evidently, this never occurs in nature. All the free vibrations die down after a time due to damping.

Consider a mass m attached with a spring of stiffness k and a viscous damper of *damping coefficient* (c) [Fig. 4.14].

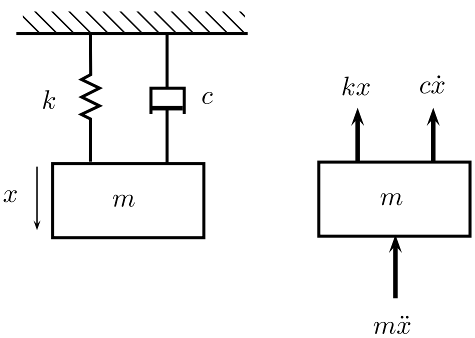


Figure 4.14 | Free damped system.

The body is displaced by distance x vertically downward from its equilibrium position. Using *Newton's method*, the equilibrium equation for the system is written as

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{4.11}$$

The solution of this equation can be in the following form:

$$x = e^{ut}$$

where u is a constant. Velocity and acceleration functions are written as

$$\dot{x} = ue^{ut}, \quad \ddot{x} = u^2 e^{ut}$$

Putting these values in Eq. (4.11) gives the following values of the constant:

$$u_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

Therefore, solution of Eq. (4.11) is written as

$$x = A_1 e^{u_1 t} + A_2 e^{u_2 t} \tag{4.12}$$

The *critical damping coefficient* c_c is the value of damping coefficient c for which

$$\left(\frac{c_c}{2m}\right)^2 - \frac{k}{m} = 0$$
$$c_c = 2\sqrt{km}$$

The ratio of c and c_c is termed as *damping ratio*, denoted by ξ:

$$\xi = \frac{c}{c_c}$$

Therefore, Eq. (4.12) can be written in terms of ξ as

$$x = A_1 e^{\left(-\xi+\sqrt{\xi^2-1}\right)\omega_n t} + A_2 e^{\left(-\xi-\sqrt{\xi^2-1}\right)\omega_n t} \tag{4.13}$$

where

$$\omega_n = \sqrt{\frac{k}{m}}$$

is the *natural frequency* of undamped free vibrations in the same system. This can be written in terms of c_c as

$$\omega_n = \frac{c_c}{2m}$$

In Eq. (4.13), the displacement (x) consists of two exponential functions of damping ratio ξ, which can be positive, negative or zero. Depending upon the value of ξ w.r.t. unity, free-damped vibration systems are classified into following:

1. Over-damped system ($\xi > 1$)
2. Critically damped system ($\xi = 1$)
3. Under-damped system ($\xi < 1$)

These are discussed in the following subsections.

4.3.1 Over-Damped System

When $\xi > 1$, the system is called *over-damped*. Using Eq. (4.13), the displacement function is re-written as

$$x = \underbrace{A_1 e^{\left(-\xi+\sqrt{\xi^2-1}\right)\omega_n t}}_{x_1} + \underbrace{A_2 e^{\left(-\xi-\sqrt{\xi^2-1}\right)\omega_n t}}_{x_2}$$

This expression contains two exponential functions, x_1 and x_2, with negative power of e; both the elements decrease exponentially with time. Therefore, the motion is aperiodic or non-oscillatory [Fig. 4.15].

The value of arbitrary constants A_1 and A_2 can be found for initial condition ($t = 0$) when the displacement and velocity are equal to $x(0)$ and $\dot{x}\,(0)$. Once the system is disturbed, it will take infinite time to come back to equilibrium position.

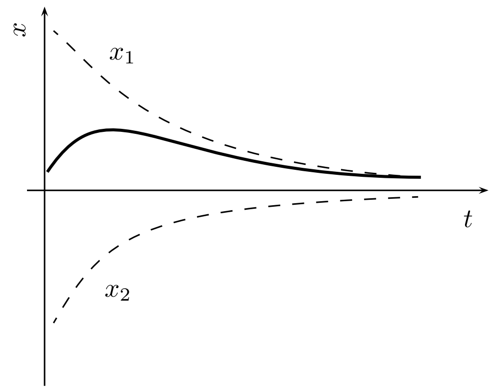


Figure 4.15 Aperiodic motion ($\xi > 1$).

4.3.2 Critically Damped System

When $\xi = 1$, the system is called to be *critically damped*. Using Eq. (4.13), the displacement function for this case is written as

$$x = (A_1 + A_2 t)\, e^{-\omega_n t}$$

where A_1 and A_2 are the arbitrary constants which can be determined from the initial conditions. This is an exponential function with negative power of e; the displacement decreases exponentially with time. Therefore, the motion is aperiodic or non-oscillatory. Figure 4.16 shows three different patterns of the function which depend upon the direction and value of initial velocity $\dot{x}(0)$, evident through the arbitrary constants A_1 and A_2.

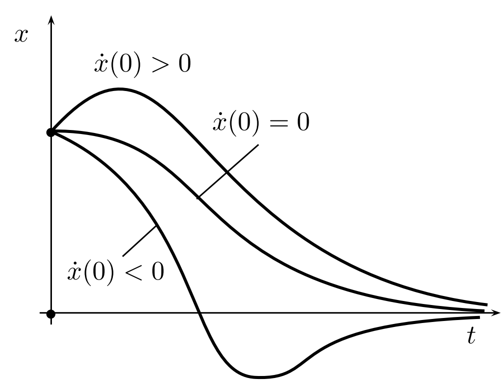


Figure 4.16 Aperiodic motion ($\xi = 1$).

The situation of critical damping measures the relative amount of damping in a particular system. Critical damping means that the amount of damping which will make the system stop vibrating within the least possible time.

4.3.3 Under-Damped System

When $\xi < 1$, the system is called to be under-damped. Using Eq. (4.13), the displacement function for this case is written as

$$x = e^{-\xi\omega_n t}\left[A_1 e^{j\sqrt{1-\xi^2}\omega_n t} + A_2 e^{-j\sqrt{1-\xi^2}\omega_n t}\right]$$

Using $e^{jx} = \cos x + j \sin x$,

$$x = A e^{-\xi\omega_n t} \sin(\omega_d t + \phi) \tag{4.14}$$

where

$$\omega_d = \sqrt{1-\xi^2}\omega_n$$

is the *frequency of damped vibrations*, ϕ is the phase difference, and A is the amplitude.

The displacement function is a multiplication of exponentially decreasing amplitude and a sinusoidal component; the motion is periodic with frequency w_d but the amplitude decreases exponentially in every cycle [Fig. 4.17].

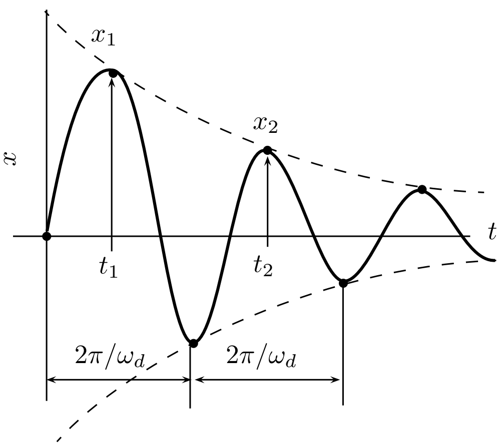


Figure 4.17 Aperiodic motion ($\xi < 1$).

To evaluate the cyclic decrement, let t_1 and t_2 $(= t_1 + t_d)$ denote the times corresponding to two successive amplitudes x_1 and x_2, respectively. Here, t_d is the time period given by

$$t_d = \frac{2\pi}{\omega_d}$$

Using Eq. (4.14), the ratio of two successive amplitudes is

$$\begin{aligned}\frac{x_1}{x_2} &= \frac{Ae^{-\xi\omega_n t_1}\sin(\omega_d t_1+\phi)}{Ae^{-\xi\omega_n t_2}\sin(\omega_d t_2+\phi)}\\ &= e^{-\xi\omega_n(t_1-t_2)}\frac{\sin(\omega_d t_1+\phi)}{\sin(\omega_d t_1+2\pi+\phi)}\\ &= e^{\xi\omega_n(t_2-t_1)}\\ &= e^{\xi\omega_n t_d}\\ \frac{x_1}{x_2} &= e^{\frac{2\pi\xi}{\sqrt{1-\xi^2}}}\end{aligned} \tag{4.15}$$

In this context, a term *logarithmic decrement* (δ) is defined as the natural logarithm of the ratio of any two successive amplitudes on the same side of the mean line:

$$\delta = \ln\left(\frac{x_1}{x_2}\right) \tag{4.16}$$

If the system executes n cycles, the ratio of amplitudes can be expressed as

$$\delta = \frac{1}{n}\ln\left(\frac{x_1}{x_n}\right)$$

Using Eqs. (4.15) and (4.16), the logarithmic decrement δ is related to ξ as

$$\delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}} \tag{4.17}$$

Using Eq. (4.17), the damping factor can be presented in terms of δ as

$$\xi = \frac{\delta}{\sqrt{4\pi^2+\delta^2}} \tag{4.18}$$

The amplitude, frequency (ω_d), and logarithmic decrement (δ) in damped vibrations depend upon the damping factor ξ; The amplitude decreases with increase in the amount of damping or ξ. Figure 4.18 shows the variation of ω_d/ω_n and δ with respect to ξ.

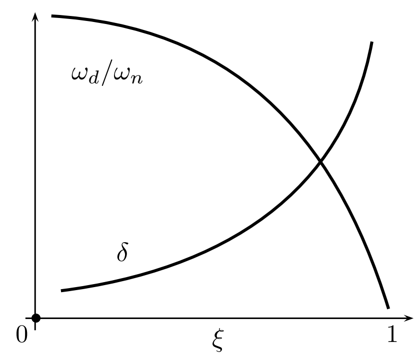


Figure 4.18 ω_d/ω_n and δ with respect to ξ.

4.4 FORCED VIBRATION

Forced vibrations take place under the excitation of external forces. The oscillations in machines, such as engines, rotating unbalance, are typical examples of forced vibrations. Forced vibrations can also be modeled as a spring–mass damper system with an external dynamic force. Based on this, the vibrations due to rotating unbalance and support excitation can also be studied.

4.4.1 Spring–Mass–Damper System

Consider a mass m, attached with spring of stiffness k and a viscous damper of coefficient c, is subjected to a dynamic force F [Fig. 4.19].

The body is displaced x distance from its equilibrium position vertically downwards. For this system, the differential equation of equilibrium is written as

$$m\ddot{x} + c\dot{x} + kx = F_0 \sin\omega t \tag{4.19}$$

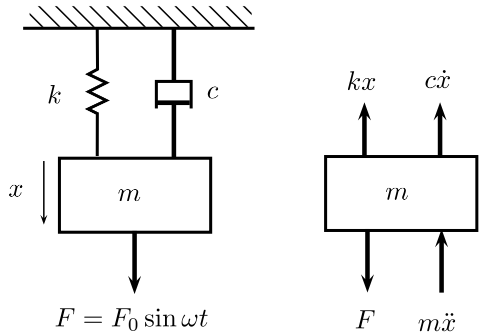


Figure 4.19 | Forced vibration.

The solution of above equation has two components, complementary function (x_c) and particular integral (x_p):

$$x = x_c + x_p$$

1. Complementary Function This component is the solution of left-hand side of Eq. (4.19) without force function $F_0 \sin \omega t$; a second order differential equation:

$$x_c = Ae^{-\xi \omega_n t} \sin \left(\sqrt{1 - \xi^2} \omega_n t + \phi_n \right)$$

2. Particular Integral This is the steady-state component of the solution. Because the force function is sinusoidal, the particular integral should also be a sinusoidal function in the following form

$$x_p = x_0 \sin (\omega t - \phi) \tag{4.20}$$

where x_0 is the amplitude, and ϕ is the phase difference by which the displacement lags the vector force.

A damped vibration dies down rapidly with time, instantaneously or slowly, depending upon the amount of damping. Therefore, the solution of Eq. (4.19) consists of only particular integral, given by Eq. (4.20), as the steady-state solution:

$$x = x_0 \sin (\omega t - \phi) \tag{4.21}$$

The velocity and acceleration of this solution are as follows:

1. Velocity Differentiating Eq. (4.21) w.r.t. time,

$$\begin{aligned} \dot{x} &= \omega x_0 \cos (\omega t - \phi) \\ &= \omega x_0 \sin \left(\omega t - \phi + \frac{\pi}{2} \right) \end{aligned} \tag{4.22}$$

2. Acceleration Differentiating Eq. (4.22) w.r.t. time,

$$\begin{aligned} \ddot{x} &= \omega^2 x_0 \cos \left(\omega t - \phi + \frac{\pi}{2} \right) \\ &= -\omega^2 x_0 \sin (\omega t - \phi + \pi) \end{aligned} \tag{4.23}$$

Thus, the functions $\dot{x}$ and $\ddot{x}$ are ahead of the displacement x by $\pi/2$ and π radians, respectively. Using Eqs. (4.21)–(4.23), the forces acting on the system (i.e., damping force, inertia force, spring force, external force) can be shown on a phase diagram [Fig. 4.20].

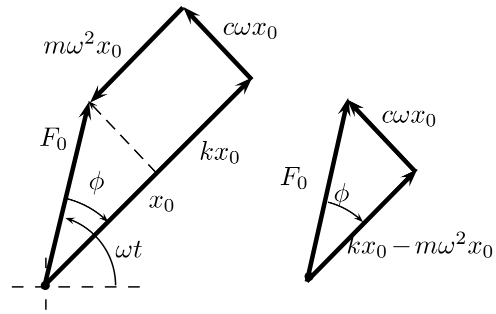


Figure 4.20 | Phase diagram in forced vibrations.

The phase diagram provides the expression of steady-state amplitude (x_0) and phase angle (ϕ) in Eq. (4.21):

1. Amplitude Using Pythagoras' theorem for the triangle of forces:

$${F_0}^2 = (c\omega x_0)^2 + \left(x_0 k - m\omega^2 x_0 \right)^2$$

The amplitude of the forced vibration is found as

$$\begin{aligned} x_0 &= \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} \\ &= \frac{x_{st}}{\sqrt{\left(1 - \omega^2/{\omega_n}^2\right)^2 + (2\xi\omega/\omega_n)^2}} \end{aligned} \tag{4.24}$$

where $x_{st} = F_0/k$ is the static displacement caused by a static force F_0 in the absence of damper.

2. Phase Angle The phase angle ϕ between force and displacement vectors is found as

$$\begin{aligned} \tan \phi &= \frac{c\omega}{k - m\omega^2} \\ &= \frac{2\xi\omega/\omega_n}{1 - \omega^2/{\omega_n}^2} \end{aligned} \tag{4.25}$$

The ratio x_0/x_{st} is known as *magnitification factor* Λ:

$$\begin{aligned} \Lambda &= \frac{x_0}{x_{st}} \\ &= \frac{1}{\sqrt{\left(1 - \omega^2/{\omega_n}^2\right)^2 + (2\xi\omega/\omega_n)^2}} \end{aligned} \tag{4.26}$$

Equations (4.24) and (4.25) can be used to plot x/x_{st} and ϕ w.r.t. ω/ω_n for different values of ξ [Fig. 4.21].

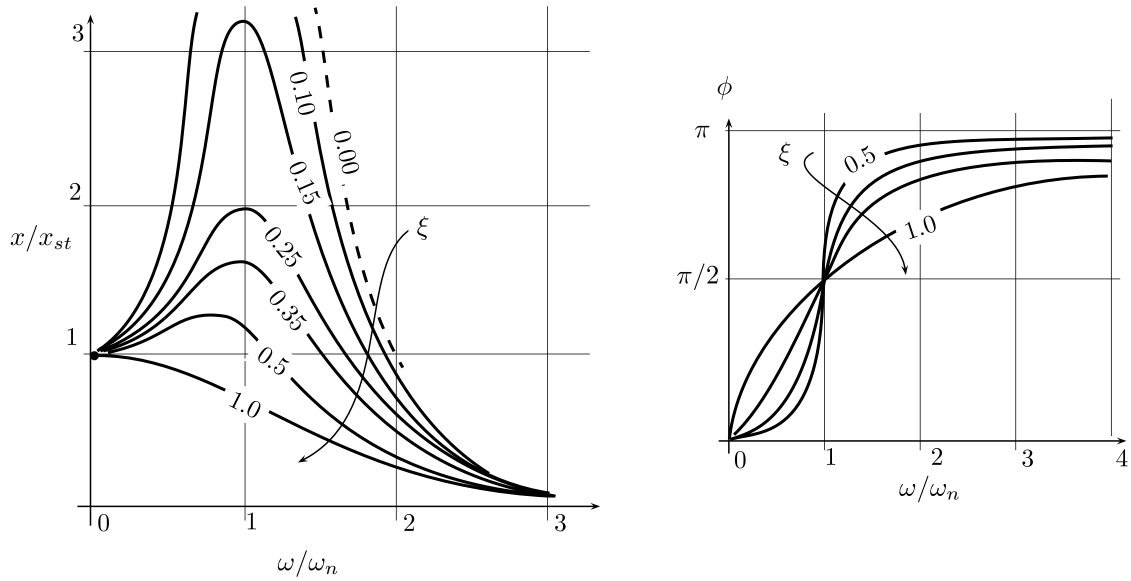


Figure 4.21 x/x_{st} and ϕ w.r.t. ω/ω_n.

The magnification factor (Λ) and phase angle (ϕ) are functions of frequency ratio and damping factor. These can be examined as follows:

1. *Effect of Frequency Ratio* The effect of frequency ratio can be examined in the following three cases:

   (a) *Small Frequencies ($\omega \ll \omega_n$)* At small frequencies ($\omega \ll \omega_n$), both inertia and damping forces are small because velocity ($\dot{x}$) and acceleration ($\ddot{x}$) are very small. The case is equivalent when the system is subjected to a static load F, which is balanced by the spring force. The amplitude and phase angle are as follows:

   $$\Lambda \approx 1$$
   $$\phi \approx 0$$

   The forces in this case are

   $$c\dot{x} = 0$$
   $$m\ddot{x} = 0$$
   $$kx = F_0$$

   Thus, the magnification factor is unity. It is independent of ξ; damping coefficient has no role to play because there is no motion. In this situation, the external force is balanced by spring force.

   (b) *Resonance ($\omega = \omega_n$)* If the frequency of the external force ω coincides with the natural frequencies (ω_n) of the system, a condition known as *resonance* occurs. The corresponding values are as follows:

   $$\Lambda = \frac{1}{2\xi}$$
   $$\phi_r = \frac{\pi}{2}$$

   In this case,

   $$c\dot{x} = F$$
   $$m\ddot{x} = kx$$

   The external force is balanced by the damping force while the inertia force is balanced by the spring force.

   (c) *Large Frequencies ($\omega \gg \omega_n$)* At large frequencies ($\omega \gg \omega_n$),

   $$\Lambda \approx 0$$
   $$\phi \approx \pi$$

   In this case,

   $$c\dot{x} = 0$$
   $$m\ddot{x} = F$$
   $$kx = 0$$

   The external force is balanced by the inertia force. Damping has no effect on the system.

2. *Effect of Damping Factor* It is interesting to find the frequency ratio $r = \omega/\omega_n$ for the peak value of the magnification factor Λ. For this, the

denominator of Eq. (4.26) should be minimum. This can be obtained by

$$\frac{d}{dr}\left\{\sqrt{\left(1-\frac{\omega^2}{{\omega_n}^2}\right)^2+\left(2\xi\frac{\omega}{\omega_n}\right)^2}\right\}=0$$

This results in

$$\frac{\omega}{\omega_n}=\sqrt{1-2\xi^2}$$

Thus, the maximum value of Λ occurs for $0<\xi<1/\sqrt{2}$. The value of frequency ratio cannot be more than 1, therefore, the peak values of Λ for different values of ξ occurs for $\omega<\omega_1$. The corresponding value of Λ is found as

$$\begin{aligned}\Lambda_{max}&=\frac{1}{\sqrt{(1-\omega^2/{\omega_n}^2)^2+(2\xi\omega/\omega_n)^2}}\\&=\frac{1}{\sqrt{(1-1-2\xi^2)^2+4\xi^2\,(1-2\xi^2)}}\\&=\frac{1}{\sqrt{4\xi^4+4\xi^2-8\xi^4}}\\&=\frac{1}{2\xi\sqrt{1-2\xi^2}}\end{aligned}$$

This is the peak value of Λ which occurs at frequency ratio $\omega/\omega_n=\sqrt{1-2\xi^2}$.

4.4.2 Rotating Unbalance

Rotating machines are not supposed to have any unbalanced mass because it induces vibrations in the system. Consider a *rotating unbalance* mass m_e at eccentric radius e with constant angular speed ω. The machine is supported by a spring–mass–damper system of stiffness k and damping coefficient c [Fig. 4.22]

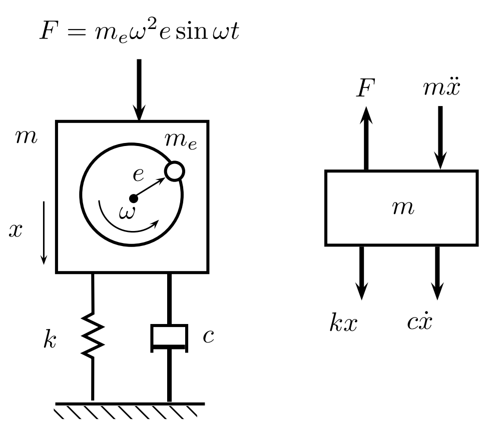


Figure 4.22 Rotating unbalance.

The unbalanced dynamic force acting on the mass m shall be written as

$$F=m_e\omega^2 e\sin\omega t$$

The model of forced vibrations in spring-mass-damper system [*Section* 4.4.1] can be applied to this system by taking the equivalent amplitude of force

$$F_0=m_e\omega^2 e$$

Accordingly, the amplitude and phase angle are found as follows:

1. *Amplitude*

$$x_0=\frac{m_e e\omega^2}{\sqrt{(k-m\omega^2)^2+(c\omega)^2}}$$

$$\frac{x_0}{x_{st}}=\frac{\omega^2/{\omega_n}^2}{\sqrt{(1-\omega^2/{\omega_n}^2)^2+(2\xi\omega/\omega_n)^2}} \quad (4.27)$$

where $x_{st}=m_e\cdot e/m$.

2. *Phase Angle* The phase angle ϕ between the force and the displacement vectors is found as

$$\tan\phi=\frac{c\omega}{k-m\omega^2}=\frac{2\xi\omega/\omega_n}{1-\ \omega^2/{\omega_n}^2} \quad (4.28)$$

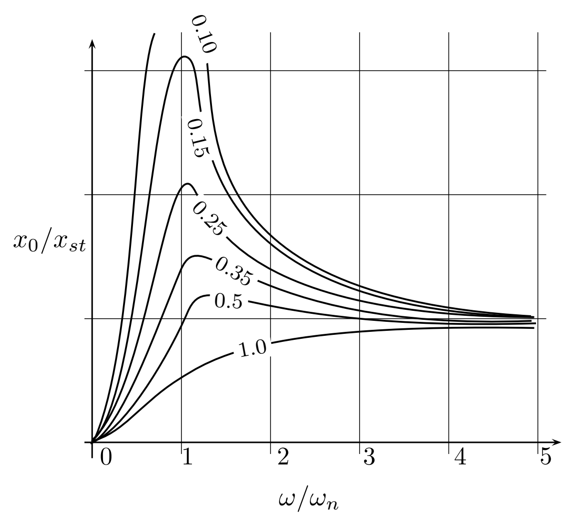


Figure 4.23 x/x_{st} for rotating unbalance.

The effect of ω/ω_n and ξ on x/x_{st} can be observed using Eqs. (4.27) and (4.28) [Fig. 4.23]:

1. *Effect of Frequency Ratio* The amplitude of all the curves is zero at $\omega = 0$, but markedly high near resonance ($\omega = \omega_n$). At higher frequencies, the amplitude ratio is almost unity where the effect of damping is negligible.

2. *Effect of Damping Factor* The peak value of amplitude is found when

$$\frac{\omega}{\omega_n} = \frac{1}{\sqrt{1-2\xi^2}}$$

Therefore, the peak value occurs for $0 < \xi < 1/\sqrt{2}$. The denominator of the above frequency ratio cannot be more than unity, therefore, the frequency ratio is always greater than 1, therefore, peak value occurs at spead ratio near and higher than 1. The corresponding peak value of the amplitude is found as

$$(x_0)_{max} = \frac{m_e e/m}{2\xi\sqrt{1-\xi^2}}$$

4.4.3 Support Excitation

Figure 4.24 shows a spring-mass-damper system in which the support of the system itself vibrates with the following displacement equation:

$$y = y_0 \sin(\omega t + \alpha)$$

The relative displacement on the spring and damper is $x - y$. Therefore, the differential equation of equilibrium can be written as

$$m\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) = 0$$
$$m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky$$

This can be written as

$$m\ddot{x} + c\dot{x} + kx = y_0\sqrt{k^2 + (c\omega)^2}\sin(\omega t + \phi)$$

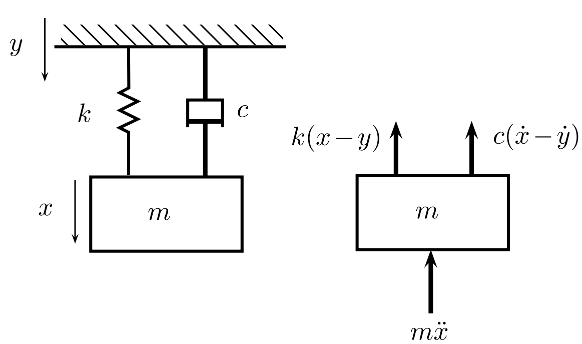


Figure 4.24 Support excitation.

The model of forced vibrations in spring-mass-damper system [*Section* 4.4.1] can be applied to this system by taking the equivalent amplitude of force

$$F_0 = y_0\sqrt{k^2 + (c\omega)^2}$$

The amplitude and phase angle are found as follows:

1. *Amplitude*

$$x_0 = \frac{y_0\sqrt{k^2 + (c\omega)^2}}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$$

$$\frac{x_0}{y_0} = \frac{\sqrt{1 + (2\xi\omega/\omega_n)^2}}{\sqrt{(1 - \omega^2/{\omega_n}^2)^2 + (2\xi\omega/\omega_n)^2}} \quad (4.29)$$

2. *Phase Angle* The phase angle ϕ force and displacement vectors is found as

$$\tan\phi = \frac{c\omega}{k - m\omega^2} = \frac{2\xi\omega/\omega_n}{1 - \omega^2/{\omega_n}^2} \quad (4.30)$$

4.4.4 Transmissibility

Machines are isolated from undesired vibrations by mounting the machines on springs and providing dash-pot mechanisms such as shock absorbers in motor cycle and automobiles.

For a system with unbalance force due to rotating mass [Fig. 4.22], the force transmitted (F_{tr}) to the base or foundation is the sum of spring force and the dashpot force:

$$F_{tr} = \sqrt{(kx)^2 + (c\omega x)^2} = x\sqrt{k^2 + c^2\omega^2}$$

The ratio of transmitted force and dynamic force is known as *transmissibility* (Tr). For the present case,

$$\mathrm{Tr} = \frac{F_{tr}}{F_0} = \frac{\sqrt{1 + (2\xi\omega/\omega_n)^2}}{\sqrt{(1 - \omega^2/{\omega_n}^2)^2 + (2\xi\omega/\omega_n)^2}}$$

This expression can be used to investigate the effect of system characteristics on the transmissibility by plotting Tr w.r.t. ω/ω_n for different values of ξ [Fig. 4.25].

The purpose of providing the spring and dampers to the machine is to make the force transmitted less than

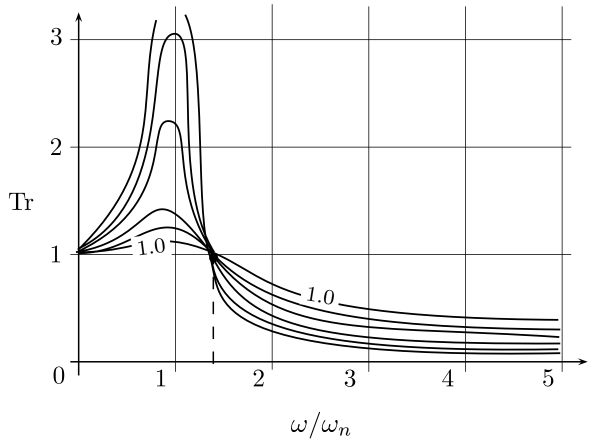


Figure 4.25 Tr versus ω/ω_n.

the impressed or periodic force; the body is isolated from large accelerations of the base when Tr < 1. The value of Tr depends upon the ratio ω/ω_n and ξ. Therefore, the effects of ω/ω_n and ξ on Tr are examined as follows:

1. *Effect of Frequency Ratio* The transmissibility curve for different damping factors have the same value Tr $= 1$ for $\omega/\omega_n = \sqrt{2}$. The isolation is achieved (Tr < 1.0) when $\omega/\omega_n > \sqrt{2}$. The system is dangerous if $\omega/\omega_n < \sqrt{2}$. It follows that ω/ω_n must be as large as possible for the required stiffness of the spring.

2. *Effect of Damping Factor* For $0 < \xi < 1$ and $\omega < \omega_n$, the maximum value of Tr is obtained at

$$\frac{\omega}{\omega_n} = \frac{1}{2\xi}\left(\sqrt{1+8\xi^2}-1\right)^{1/2}$$

At this frequency, the peak value of Tr is obtained as

$$\text{Tr}_m = 4\xi^2\sqrt{\frac{\sqrt{1+8\xi^2}}{2+16\xi^2+(16\xi^4-8\xi^2-2)\sqrt{1+8\xi^2}}}$$

If $\xi = 0$, then transmissibility is written as

$$\text{Tr} = \frac{1}{1-(\omega/\omega_n)^2}$$

4.4.5 Whirling of Rotating Shafts

When a rotor is mounted on a shaft, its center of mass does not usually coincide with the center line of the shaft. Therefore, when the shaft rotates, it is subjected to a centrifugal force which makes the shaft to bend in the direction of eccentricity of the center of mass. The shaft tends to bow out at certain speed and whirl in a complicated manner. This increases the eccentricity of the mass, and hence the centrifugal force. In this way, the effect is cumulative and ultimately the shaft can even fail.

Critical speed or *whirling speed* is the speed at which the shaft tends to vibrate violently in transverse direction. This is also called *whipping speed*. In general, the critical speeds of any circular shaft coincide with the natural frequencies of vibrations of the non-rotating shaft on its bearings. Below the critical speeds, the shaft offers some elastic resistance to a sidewise force, and this is no longer true at the critical speed. It has been observed that at critical speed, the shaft again becomes almost straight. But at some other speed, the same phenomenon recurs, the only difference being that the shaft now bends in two bows and so on.

Consider a rotor of mass m assembled on a shaft of stiffness k with an eccentricity e. The shaft rotates with angular velocity ω and rotor gets additional deflection due to centrifugal force [Fig. 4.26].

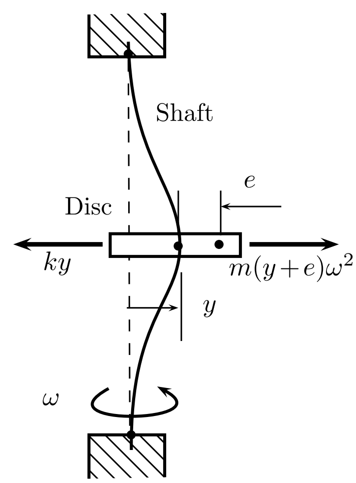


Figure 4.26 Critical speed of shaft.

The unbalanced mass is in equilibrium under the centrifugal force $m\,(y+e)\,\omega^2$ and force resisting the deflection ky:

$$ky = m\,(y+e)\,\omega^2$$

$$y = \frac{e}{(\omega/\omega_n)^2-1} \tag{4.31}$$

where ω_n $(= \sqrt{k/m})$ is the natural frequency. From Eq. (4.31), when $\omega = \omega_n$, the deflection y is infinitely large (resonance occurs) and the speed ω is the critical speed. By increasing the frequency ω beyond ω_n, y approaches the value $-e$ or the center of mass of the rotor approaches the center line of the rotation. This principle is used in running high-speed turbines by speeding up the rotor rapidly beyond the critical speed and the rotor runs steadily.

IMPORTANT FORMULAS

Fundamentals

1. Harmonic motion

$$x = x_0 \sin(\omega t + \phi)$$

$$\bar{x} = \lim_{T\to\infty} \frac{1}{T}\int_0^T x(t)\,dt$$

$$\bar{x}^2 = \lim_{T\to\infty} \frac{1}{T}\int_0^T x^2(t)\,dt$$

$$f = \frac{1}{T}\ \text{Hz}$$

$$f = \frac{\omega}{2\pi}\ \text{cycles/s}$$

$$= \omega\ \text{rad/s}$$

$$= \frac{60\omega}{2\pi}\ \text{rpm}$$

2. Work done per cycle

$$W = \pi F_o x_o \sin\phi$$

3. Spring

$$U(x) = \frac{1}{2}kx^2$$

$$k_e = k_1 + k_2 \quad \text{Parallel}$$

$$\frac{1}{k_e} = \frac{1}{k_1} + \frac{1}{k_2} \quad \text{Series}$$

4. Damping coefficient

$$F = c\dot{x}$$

$$E = \oint F\cdot dx = \pi c\omega x_0^2$$

$$\beta = \frac{E}{U}, \quad \eta = \frac{E}{2\pi U}$$

5. Coulomb damping

$$\Delta = \frac{4F}{k}$$

6. Structural damping

$$\eta = \frac{E}{2\pi U}$$

$$c_e = \frac{\alpha}{\pi\omega} = \frac{\alpha}{\pi k}$$

Undamped Free Vibrations

$$m\ddot{x} + kx = 0$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

1. Energy method

$$\frac{d}{dt}(T+U) = 0$$

2. Rayleighs energy method

$$T_{max} = U_{max}$$

3. Cylinder on cylindrical face

$$\omega_n = \sqrt{\frac{2g}{3(R-r)}}$$

4. Pendulum

$$\omega = \sqrt{\frac{g}{l}}$$

Free Damped Vibrations

$$m\ddot{x} + c\dot{x} + kx = 0$$

$$c_c = 2\sqrt{km}, \quad \xi = \frac{c}{c_c}, \quad \omega_n = \frac{c_c}{2m}$$

$$\omega_d = \sqrt{1-\xi^2}\,\omega_n$$

$$\delta = \ln\left(\frac{x_1}{x_2}\right) = \frac{1}{n}\ln\left(\frac{x_1}{x_n}\right)$$

$$= \frac{2\pi\xi}{\sqrt{1-\xi^2}}$$

$$\xi = \frac{\delta}{\sqrt{4\pi^2+\delta^2}}$$

Forced Vibrations

$$m\ddot{x} + c\dot{x} + kx = F_0 \sin\omega t$$

$$x_0 = \frac{F_0}{\sqrt{(k-m\omega^2)^2 + (c\omega)^2}}$$

$$= \frac{x_{st}}{\sqrt{(1-\omega^2/\omega_n{}^2)^2 + (2\xi\omega/\omega_n)^2}}$$

$$\tan\phi = \frac{c\omega}{k-m\omega^2} = \frac{2\xi\omega/\omega_n}{1-\omega^2/\omega_n{}^2}$$

$$\Lambda_{max} = \frac{1}{2\xi\sqrt{1-2\xi^2}}$$

$$\text{when } \frac{\omega}{\omega_n} = \sqrt{1-2\xi^2}$$

Rotating Unbalance

$$F = m_e\omega^2 e \sin\omega t$$

$$F_0 = m_e\omega^2 e$$

$$\frac{x_0}{x_{st}} = \frac{\omega^2/\omega_n{}^2}{\sqrt{(1-\omega^2/\omega_n{}^2)^2 + (2\xi\omega/\omega_n)^2}}$$

$$\tan\phi = \frac{c\omega}{k-m\omega^2}$$

$$= \frac{2\xi\omega/\omega_n}{1-\ \omega^2/\omega_n{}^2}$$

$$(x_0)_{max} = \frac{m_e e/m}{2\xi\sqrt{1-\xi^2}}$$

$$\text{when } \frac{\omega}{\omega_n} = \frac{1}{\sqrt{1-2\xi^2}}$$

Support Excitation

$$m\ddot{x} + c(\dot{x}-\dot{y}) + k(x-y) = 0$$

$$\frac{x_0}{y_0} = \frac{\sqrt{1+(2\xi\omega/\omega_n)^2}}{\sqrt{(1-\omega^2/\omega_n{}^2)^2 + (2\xi\omega/\omega_n)^2}}$$

$$\tan\phi = \frac{c\omega}{k-m\omega^2}$$

$$= \frac{2\xi\omega/\omega_n}{1-\omega^2/\omega_n{}^2}$$

Transmissibility

$$\text{Tr} = \frac{\sqrt{1+(2\xi\omega/\omega_n)^2}}{\sqrt{(1-\omega^2/\omega_n{}^2)^2 + (2\xi\omega/\omega_n)^2}}$$

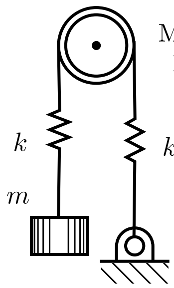
Critical Speed of Shaft

$$y = \frac{e}{(\omega/\omega_n)^2 - 1}$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

SOLVED EXAMPLES

1. An extensible string of stiffness k in each side of the massless pulley supports a mass m:



Determine the natural frequency of the system.

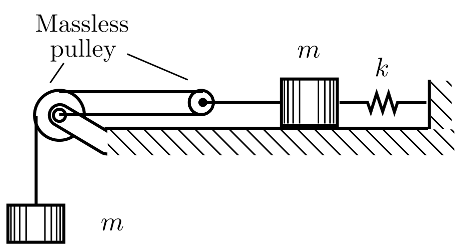
Solution. The system can be modeled as equivalent spring-mass system having stiffness (springs in series):

$$\frac{1}{k_e} = \frac{1}{k} + \frac{1}{k}$$
$$k_e = \frac{k}{2}$$

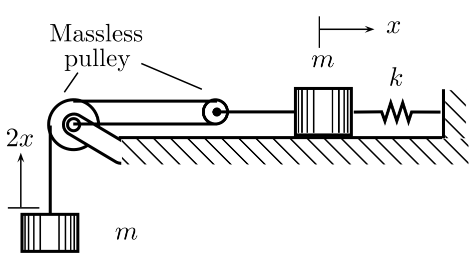
Therefore, the natural frequency of the system is

$$\omega = \sqrt{\frac{k}{2m}}$$

2. Determine the natural frequency of the system.



Solution. Let T be the tension in the string. The system has dependent motion. For movement x of mass on the plane, the mass over the pulley shall move by $2x$:



The spring is under static extension δ_{st} given by

$$\delta_{st} k = T_{st} = 2mg$$

Therefore, the respective equations of motion can be written as

$$m\ddot{x} + k\left(x + \delta_{st}\right) - 2T = 0$$
$$m \times 2\ddot{x} + T - mg = 0$$

Eliminating T from above equations, one finds

$$5m\ddot{x} + kx = 0$$

Therefore, comparing with spring-mass model:

$$\omega = \sqrt{\frac{k}{5m}}$$

3. A simple U tube manometer is filled with liquid of specific gravity s. The cross-sectional area of tube is a and length of the liquid column is l. Determine the natural frequency of oscillations of the liquid column. If the value of length of the column is 20 cm, what will be the natural frequency of oscillations?

Solution. For a displacement x, the total energy of the system is given by

$$\frac{1}{2}\rho a l \dot{x}^2 + (\rho a g x)\, x = 0$$

Therefore, differentiating w.r.t. t on both sides,

$$\rho a l \dot{x}\ddot{x} + 2\rho a g x \dot{x} = 0$$
$$\rho a l \ddot{x} + 2\rho a g x = 0$$

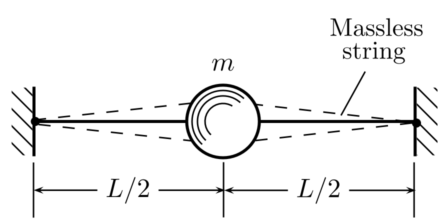
Hence, the natural frequency of oscillations is

$$f = \frac{1}{2\pi}\sqrt{\frac{2g}{l}}$$

For $l = 0.2$ m,

$$\omega = 1.57 \text{ Hz}$$

4. A light ball of mass m is tightly stretched by two strings with initial tension T:



Determine the natural frequency of the ball if it is plucked vertically to a small distance.

Solution. Let the ball is given a slight angular deflection (θ) in vertical position. The ball is vertically displaced by

$$\theta = \frac{x}{L/2}$$

The ball will be under equilibrium of inertia force and resolved tension on both the strings (taking $\sin\theta \approx \theta$):

$$m\ddot{x} + 2T\theta = 0$$
$$m\ddot{x} + 2T \times \frac{x}{L/2} = 0$$
$$m\ddot{x} + \frac{4T}{L} \times x = 0$$

Therefore,

$$\omega = \sqrt{\frac{4T}{mL}}$$

5. A 10 kg mass is supported on a spring of stiffness 4 kN/m and has a dash pot which produces a resistance of 20 N at velocity of 0.25 m/s. Determine the natural frequency and damping ratio of the system.

Solution. Given that

$$m = 10 \text{ kg}$$
$$k = 4 \times 10^3 \text{ N/m}$$

Natural frequency of the system is

$$\omega_n = \sqrt{\frac{k}{m}} = 20 \text{ rad/s}$$

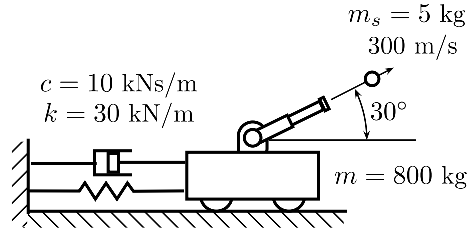
Damping coefficient is

$$c = \frac{20}{0.25} = 80 \text{ Ns/m}$$

The damping ratio is determined as

$$\xi = \frac{c}{c_c} = \frac{c}{2\sqrt{km}} = 0.2$$

6. A gun-carrying vehicle fires a shell of mass 5 kg at speed 300 m/s inclined at 30° from the horizontal. The combined mass of the gun and the vehicle is 800 kg. The recoil mechanism is critically damped and has an equivalent stiffness of 30 kN/m.



Determine the maximum recoil of the gun-vehicle unit.

Solution. Given that

$$m_s = 5 \text{ kg}$$
$$v = 300 \text{ m/s}$$
$$\theta = 20^\circ$$

For gun-vehicle system,

$$m = 800 \text{ kg}$$
$$k = 30 \times 10^3 \text{ N/m}$$
$$c = 9.8 \times 10^3 \text{ Ns/m}$$

Natural frequency of the system is

$$\omega_n = \sqrt{\frac{k}{m}} = 6.12 \text{ rad/s}$$

The critically damped displacement can be written as

$$x = (A_1 + A_2 t)\, e^{-\omega_n t}$$

The unknown constant A_1 and A_2 can be determined using initial conditions. Taking $x(0) = 0$,

$$A_1 = 0$$
$$A_2 = \dot{x}(0)$$

Initial recoil velocity of the gun-vehicle can be determined using the principle of conservation of linear momentum:

$$800 \times \dot{x}(0) = 5 \times 300 \times \cos 30^\circ$$
$$\dot{x}(0) = 1.62 \text{ m/s}$$

Thus, the displacement can be written as

$$x = 1.62te^{-6.12t}$$

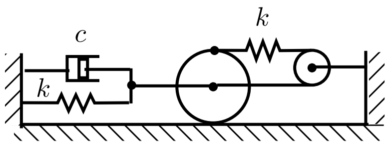
For maximum value of x,

$$t = \frac{1}{6.17} = 0.162 \text{ s}$$

Thus, the maximum displacement will be given by

$$x = 1.62 \times 0.162 \times e^{-1} = 96.5 \text{ mm}$$

7. Center of a sphere of mass m and radius r is attached to a spring-dashpot system of stiffness k and damping constant c on the left side. It is also attached to a spring of stiffness k with a string passing over pulley on the right side.



Determine the natural frequency and damping ratio of the system.

Solution. The sphere will oscillate about the bottom contact point. For slight angular deflection θ of the sphere, the spring and dashpot shall extend by $x = r\theta$. The spring attached to the string over pulley shall extend by $2r\theta$ on left side and $r\theta$ on right side, total extension $3r\theta$.

Moment of inertia of the sphere about the bottom contact point will be

$$\begin{aligned} I &= \frac{mr^2}{2} + mr^2 \\ &= \frac{3}{2}mr^2 \end{aligned}$$

Taking moments of forces about the bottom contact point

$$\begin{aligned} \frac{3mr^2}{2}\frac{\ddot{x}}{r} + r(kx + c\dot{x}) + 2r(k \times 3x) &= 0 \\ \frac{3m}{2}\ddot{x} + c\dot{x} + 10kx &= 0 \end{aligned}$$

Thus, the natural frequency of the system is

$$\begin{aligned} \omega &= \sqrt{\frac{10k}{3m/2}} \\ &= \sqrt{\frac{20k}{3m}} \end{aligned}$$

Damping ratio is

$$\begin{aligned} \xi &= \frac{c}{2\sqrt{10k \times 3m/2}} \\ &= \frac{c}{2\sqrt{15km}} \end{aligned}$$

8. A damped system has stiffness $k = 450$ kN/m and time period 2.0 s. The ratio of a consecutive amplitudes is 4.0. Determine the amplitude and phase of the steady state motion when a dynamic force $F = 2.5\cos 3t$ N acts on the system.

Solution. Given that

$$\begin{aligned} k &= 650 \text{ N/m} \\ t_d &= 2.0 \text{ s} \\ \frac{x_0}{x_1} &= 4.2 \\ F_0 &= 2.5 \text{ N} \\ \omega &= 3 \text{ rad/s} \end{aligned}$$

Using

$$\begin{aligned} \delta &= \frac{1}{n}\ln\left(\frac{x_1}{x_n}\right) \\ &= \frac{1}{1}\ln(4.2) \\ &= 1.38 \end{aligned}$$

The damping ratio can be found as

$$\begin{aligned} \xi &= \frac{\delta}{\sqrt{\delta^2 + 4\pi^2}} \\ &= 0.21 \end{aligned}$$

Damped frequency of the system is

$$\begin{aligned} \omega_n &= \frac{2\pi}{t_d} \\ &= \frac{2\pi}{t_d\sqrt{1-\zeta^2}} \\ &= 3.21 \text{ rad/s} \end{aligned}$$

Thus, frequency ratio is

$$\begin{aligned} \frac{\omega}{\omega_n} &= \frac{3}{3.2} \\ &= 0.9375 \end{aligned}$$

Static displacement is

$$\begin{aligned} x_{st} &= \frac{F_0}{k} \\ &= \frac{2.5}{650} \\ &= 3.846 \text{ mm} \end{aligned}$$

Amplitude of steady state vibrations is

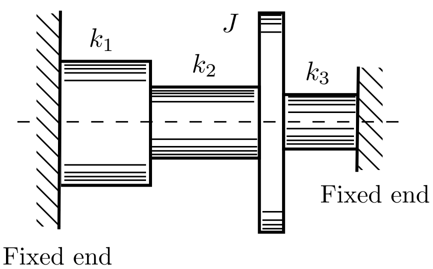
$$\begin{aligned} x_0 &= \frac{x_{st}}{\sqrt{(1-\omega^2/\omega_n{}^2)^2 + (2\xi\omega/\omega_n)^2}} \\ &= \frac{3.846}{\sqrt{0.01466 + 0.155}} \\ &= \frac{3.846}{0.4119} \\ &= 9.33 \text{ mm} \end{aligned}$$

Phase lag is

$$\begin{aligned} \tan\phi &= \frac{2\xi\omega/\omega_n}{1-(\omega^2/\omega_n{}^2)} \\ &= \frac{0.39375}{0.121} \\ &= 3.251 \\ \phi &= 72.9^\circ \end{aligned}$$

GATE PREVIOUS YEARS' QUESTIONS

1. Consider the arrangement shown in the figure below where J is the combined polar mass moment of inertia of the disc and the shafts; k_1, k_2, k_3 are the torsional stiffness of the respective shafts.



The natural frequency of torsional oscillation of the disc is given by

(a) $\sqrt{(k_1+k_2+k_3)/J}$

(b) $\sqrt{(k_1k_2+k_2k_3+k_3k_1)/(J(k_1+k_2))}$

(c) $\sqrt{(k_1k_2k_3)/(J(k_1k_2+k_2k_3+k_3k_1))}$

(d) $\sqrt{(k_1k_2+k_2k_3+k_3k_1)/(J(k_2+k_3))}$

(GATE 2003)

Solution. Equivalent setup is k_1 and k_2 in series, which is parallel to k_3, therefore, equivalent stiffness is

$$k_e = \frac{1}{1/k_1+1/k_2}+k_3 = \frac{k_1k_2+k_2k_3+k_3k_1}{k_1+k_2}$$

Natural frequency of vibrations is

$$\omega_n = \sqrt{\frac{k_e}{J}} = \sqrt{\frac{k_1k_2+k_2k_3+k_3k_1}{J(k_1+k_2)}}$$

Ans. (b)

2. A flexible rotor-shaft system comprises a 10 kg rotor disc placed in the middle of a massless shaft of diameter 30 mm and length 500 mm between bearings (shaft is being taken mass-less as the equivalent mass of the shaft is included in the rotor mass) mounted at the ends. The bearings are assumed to simulate simply supported boundary conditions. The shaft is made of steel for which the value of E is 2.1×10^{11} Pa. What is the critical speed of rotation of the shaft?

(a) 60 Hz (b) 90 Hz
(c) 135 Hz (d) 180 Hz

(GATE 2003)

Solution. Given that

$$d = 0.030 \text{ m}$$
$$l = 0.5 \text{ m}$$
$$m = 10 \text{ kg}$$
$$E = 2.1\times10^{11} \text{ Pa}$$

Moment of inertia of the shaft is

$$I = \frac{\pi d^4}{64}$$

For simply supported beams, stiffness

$$k = \frac{W}{\delta} = \frac{48l^3}{EI}$$

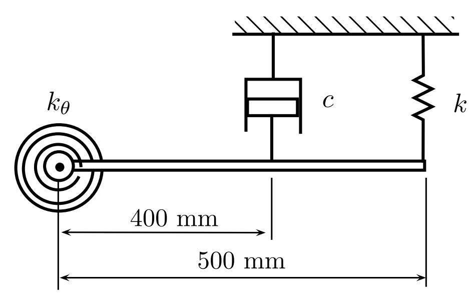
Therefore, critical speed of the shaft is

$$f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = 90.1203 \text{ Hz}$$

Ans. (b)

Common Data Questions

A uniform rigid slender bar of mass 10 kg, hinged at the left end is suspended with the help of spring and damper arrangement as shown in the figure where $k = 2$ kN/m, $C = 500$ Ns/m and the stiffness of the torsional spring k_θ is 1 kNm/rad. Ignore the hinge dimensions.



3. The undamped natural frequency of oscillations of the bar about the hinge point is

(a) 42.43 rad/s (b) 30 rad/s
(c) 17.32 rad/s (d) 14.14 rad/s

(GATE 2003)

Solution. Given that

$$m = 10 \text{ kg}$$
$$k = 2\times10^3 \text{ N/m}$$
$$k_\theta = 1\times10^3 \text{ N-m/rad}$$
$$c = 500 \text{ Ns/m}$$

For small angular displacement θ, taking torsional moments about hinge,

$$\frac{m\times0.5^2}{3}\ddot{\theta}+0.4^2\dot{\theta}c+0.5^2\theta k+\theta k_\theta = 0$$
$$\frac{m\times0.5^2}{3}\ddot{\theta}+0.4^2\dot{\theta}c+\left(0.5^2k+k_\theta\right)\theta = 0$$

Therefore, the natural frequency of vibrations is

$$\omega_n = \frac{0.5^2k+k_\theta}{m\times0.5^2/3}$$
$$= 42.4264 \text{ rad/s}$$

Ans. (a)

4. The damping coefficient in the vibration equation is given by

(a) 500 Nms/rad (b) 500 N/(m/s)
(c) 80 Nms/rad (d) 80 N/(m/s)

(GATE 2003)

Solution. Equivalent damping coefficient is

$$c_e = 0.4^2c$$
$$= 80 \text{ Ns/m}$$

Ans. (c)

5. A vibrating machine is isolated from the floor using springs. If the ratio of excitation frequency of vibration of machine to the natural frequency of the isolation system is equal to 0.5, the transmissibility of ratio of isolation is

(a) 1/2 (b) 3/4
(c) 4/3 (d) 2

(GATE 2004)

Solution. Given that

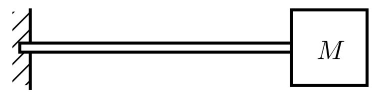
$$\xi = 0$$
$$\frac{\omega}{\omega_n} = 0.5$$

Transmissibility is determined as

$$\text{Tr} = \frac{\sqrt{1+(2\xi\omega/\omega_n)^2}}{\sqrt{(1-\omega^2/{\omega_n}^2)^2+\left(2\xi\frac{\omega}{\omega_n}\right)^2}}$$
$$= \frac{1}{1-(\omega/\omega_n)^2}$$
$$= \frac{4}{3}$$

Ans. (c)

6. A mass M of 20 kg is attached to the free end of a steel cantilever beam of length 1000 mm having a cross-section of 25×25 mm. Assume the mass of the cantilever to be negligible and E_{steel} = 200 GPa.



If the lateral vibration of this system is critically damped using a viscous damper, the damping constant of the damper is

(a) 1250 Ns/m (b) 625 Ns/m
(c) 312.50 Ns/m (d) 156.25 Ns/m

(GATE 2004)

Solution. Given that

$$M = 20 \text{ kg}$$
$$l = 1 \text{ m}$$
$$A = 0.025\times0.025 \text{ m}^2$$
$$E = 200\times10^9 \text{ Pa}$$

The moment of inertia and stiffness of cantilever is determined as

$$I = \frac{bd^4}{12}$$
$$\delta = \frac{Pl^3}{3EI}$$
$$k = \frac{W}{\delta}$$
$$= \frac{3EI}{l^3}$$

Critical damping coefficient is

$$c_c = 2\sqrt{k\times M}$$
$$= 1250 \text{ Ns/m}$$

Ans. (a)

7. A simple pendulum of length 5 m, with a bob of mass 1 kg, is in simple harmonic motion. As it passes through its mean position, the bob has a speed of 5 m/s. The net force on the bob at the mean position is

(a) zero (b) 2.5 N
(c) 5 N (d) 25 N

(GATE 2005)

Solution. At mean position, net force on the bob will be zero because acceleration is zero.

Ans. (a)

8. There are four samples P, Q, R and S, with natural frequencies 64, 96, 128 and 256 Hz, respectively. These are mounted on test setups for conducting vibration experiments. If a loud pure note of frequency 144 Hz is produced by some instrument, which of the samples will show the most perceptible induced vibration?

(a) P (b) Q
(c) R (d) S

(GATE 2005)

Solution. For most perceptible vibrations, the induced frequency should be nearer to the natural frequency.

Ans. (c)

9. In a spring-mass system, the mass is 0.1 kg and the stiffness of the spring is 1 kN/m. By introducing a damper, the frequency of oscillation is found to be 90% of the original value. What is the damping coefficient of the damper?

(a) 1.2 Ns/m (b) 3.4 Ns/m
(c) 8.7 Ns/m (d) 12.0 Ns/m

(GATE 2005)

Solution. Given that

$$m = 0.1 \text{ kg}$$
$$k = 1000 \text{ N/m}$$
$$\omega_d = 0.9\omega_n$$
$$\frac{\omega_d}{\omega_n} = 0.9$$

Therefore, critical damping coefficient is

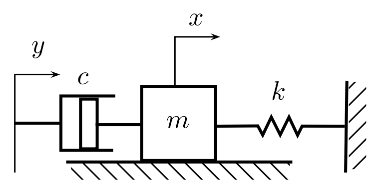
$$c_c = 2\sqrt{km} = 20 \text{ Ns/m}$$

Using,

$$\omega_d = \sqrt{1-\xi^2}\omega_n$$
$$\frac{c}{c_c} = \xi = \sqrt{1-\left(\frac{\omega_d}{\omega_n}\right)^2} = 0.43589$$
$$c = 8.7178 \text{ Ns/m}$$

Ans. (c)

10. The differential equation governing the vibrating system is:



(a) $m\ddot{x}+c\dot{x}+k(x-y)=0$
(b) $m(\ddot{x}-\ddot{y})+c(\dot{x}-\dot{y})+kx=0$
(c) $m\ddot{x}+c(\dot{x}-\dot{y})+kx=0$
(d) $m(\ddot{x}-\ddot{y})+c(\dot{x}-\dot{y})+k(x-y)=0$

(GATE 2006)

Solution. The relative motion at damper, as compared to simple spring-mass-damper system, is $(\dot{x}-\dot{y})$, therefore the equivalent differential equation for the given system is

$$m\ddot{x}+c(\dot{x}-\dot{y})+kx=0$$

Ans. (c)

11. A machine of 250 kg mass is supported on springs of total stiffness 100 kN/m. Machine has an unbalanced rotating force of 350 N at the speed of 3600 rpm. Assuming a damping factor of 0.15, the value of transmissibility ratio is:

(a) 0.0531 (b) 0.9922
(c) 0.0162 (d) 0.0028

(GATE 2006)

Solution. Given that

$$m = 250 \text{ kg}$$
$$k = 100\times 10^3 \text{ N/m}$$
$$N = 3600 \text{ rpm}$$
$$\omega = \frac{2\pi N}{60} = 376.991 \text{ rad/s}$$
$$\xi = 0.15$$

Natural frequency of vibrations in the system is

$$\omega_n = \sqrt{\frac{k}{m}} = 20 \text{ rad/s}$$

Transmissibility ratio is defined as

$$Tr = \frac{F_{tr}}{F_0} = \frac{\sqrt{1+(2\xi\omega/\omega_n)^2}}{\sqrt{(1-\omega^2/\omega_n{}^2)^2+(2\xi\omega/\omega_n)^2}} = 0.016206$$

Ans. (c)

Linked Answer Questions

A vibratory system consists of a mass 12.5 kg, a spring of stiffness 1000 N/m, and a dashpot with damping coefficient of 15 Ns/m.

12. The value of critical damping of the system is

(a) 0.223 Ns/m (b) 17.88 Ns/m
(c) 71.4 Ns/m (d) 223.6 Ns/m

(GATE 2006)

Solution. Given that

$$m = 12.5 \text{ kg}$$
$$k = 1000 \text{ N/m}$$
$$c = 15 \text{ Ns/m}$$

Therefore, critical damping coefficient is

$$c_c = 2\sqrt{km} = 2\sqrt{1000\times 12.5} = 223.607 \text{ Ns/m}$$

Ans. (d)

13. The value of logarithmic decrement is

(a) 1.35 (b) 1.32
(c) 0.68 (d) 0.66

(GATE 2006)

Solution. Damping factor is

$$\xi = \frac{c}{c_c} = 0.067082$$

Logarithmic increment is

$$\delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}} = 0.42244$$

Ans. (d)

14. For an under-damped harmonic oscillator, resonance

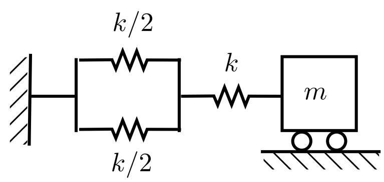
(a) occurs when excitation frequency is greater than undamped natural frequency
(b) occurs when excitation frequency is less than undamped natural frequency
(c) occurs when excitation frequency is equal to undamped natural frequency
(d) never occurs

(GATE 2007)

Solution. In under-damped vibrations, $\xi < 1$, and in such cases, vibrations can not find any probability of resonance.

Ans. (d)

15. The natural frequency of the system shown below is



(a) $\sqrt{k/(2m)}$ (b) $\sqrt{k/m}$
(c) $\sqrt{2k/m}$ (d) $\sqrt{3k/m}$

(GATE 2007)

Solution. The equivalent spring constant of the parallel springs is

$$k_e = 2\times\frac{k}{2} = k$$

Therefore, it constitutes two springs of stiffness k in series, therefore, equivalent spring constant is $k/2$. Hence, the natural frequency is given by

$$\omega_n = \sqrt{\frac{k}{2m}}$$

Ans. (a)

16. The equation of motion of a harmonic oscillator is given by

$$\frac{d^2x}{dt^2}+2\xi\omega_n\frac{dx}{dt}+\omega_n{}^2x = 0$$

and the initial condition at $t = 0$ are $x(0) = \chi$, $dx/dt(0) = 0$. The amplitude of $x(t)$ after n complete cycles is

(a) $\chi \exp\left(-2n\pi\xi/\sqrt{1-\xi^2}\right)$

(b) $\chi \exp\left(2n\pi\xi/\sqrt{1-\xi^2}\right)$

(c) $\chi \exp\left(-2n\pi\sqrt{1-\xi^2}/\xi\right)$

(d) χ

(GATE 2007)

Solution. Comparing with the equilibrium equation for general spring–mass–damper system,

$$m\ddot{x}+c\dot{x}+kx=0$$

Thus,

$$\frac{c}{m}=2\xi\omega_n, \quad \frac{k}{m}=\omega_n$$

For the above equation, the amplitude after n cycles is written as

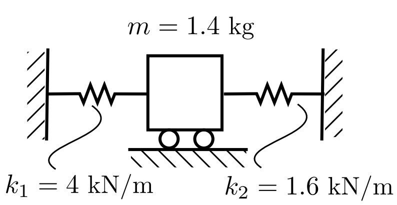
$$\delta=\frac{1}{n}\ln\frac{x_1}{x_n}=\frac{2\pi\xi}{\sqrt{1-\xi^2}}$$

Therefore,

$$\begin{aligned} x_n &= x_1 \exp(n\delta) \\ &= \chi \exp\left(2n\pi\xi/\sqrt{1-\xi^2}\right) \end{aligned}$$

Ans. (b)

17. The natural frequency of the spring–mass system shown in the figure is closest to



(a) 8 Hz (b) 10 Hz
(c) 12 Hz (d) 14 Hz

(GATE 2008)

Solution. The equivalent spring constant of two springs in parallel is

$$\begin{aligned} k_e &= k_1+k_2 \\ &= 4000+1600 \\ &= 5600 \text{ N/m} \end{aligned}$$

Therefore, the natural frequency of vibrations of the system is

$$\begin{aligned} f &= \frac{1}{2\pi}\sqrt{\frac{k}{m}} \\ &= 10.87 \text{ Hz} \end{aligned}$$

Ans. (b)

18. A uniform rigid rod of mass $m = l$ kg and length $L = 1$ m is hinged at its center and laterally supported at one end by a spring of spring constant $k = 300$ N/m. The natural frequency ω_n in rad/s is

(a) 10 (b) 20
(c) 30 (d) 40

(GATE 2008)

Solution. Given

$$\begin{aligned} k &= 300 \text{ N/m} \\ m &= 1 \text{ kg} \end{aligned}$$

Therefore, the natural frequency of vibrations of the system is

$$\begin{aligned} \omega_n &= \sqrt{\frac{k}{m/3}} \\ &= \sqrt{\frac{300}{1/3}} \\ &= 30 \text{ rad/s} \end{aligned}$$

This is because the mass of the spring is not under the uniform force, but it varies linearly from zero at corner to maximum at spring support. In such cases, equivalent mass is found to be one-third of the original mass.

Ans. (c)

19. The rotor shaft of a large electric motor supported between short bearings at both the ends shows a deflection of 1.8 mm in the middle of the rotor. Assuming the rotor to be perfectly balanced and supported at knife edges at both the ends, the likely critical speed (in rpm) of the shaft is

(a) 350 (b) 705
(c) 2810 (d) 4430

(GATE 2009)

Solution. Given that $\delta = 1.8\times10^{-3}$ m.Thus, the critical speed is given by

$$\begin{aligned} \omega_n &= \frac{60}{2\pi}\sqrt{\frac{g}{\delta}} \\ &= 704.968 \text{ rpm} \end{aligned}$$

Ans. (b)

20. An automotive engine weighing 240 kg is supported on four springs with linear characteristics. Each of the front two springs have a stiffness of 16 MN/m, while the stiffness of each rear spring is 32 MN/m. The engine speed (in rpm), at which resonance is likely to occur, is

(a) 6040 (b) 3020
(c) 1424 (d) 955

(GATE 2009)

Solution. Given that

$$m = 240 \text{ kg}$$
$$k_1 = 16 \times 10^6 \text{ N/m}$$
$$k_2 = 32 \times 10^6 \text{ N/m}$$

Therefore, the equivalent spring constant

$$k_e = 2k_1 + 2k_2 = 96 \times 10^6 \text{ N/m}$$

The natural frequency of vibration

$$w_n = \frac{60}{2\pi}\sqrt{\frac{k_e}{m}} = 6039.51 \text{ rpm}$$

Ans. (a)

21. A vehicle suspension system consists of a spring and a damper. The stiffness of the spring is 3.6 kN/m and the damping constant of the damper is 400 Ns/m. If the mass is 50 kg, then the damping factor (ξ) and damped natural frequency (ω_n), respectively, are

(a) 0.471 and 1.19 Hz
(b) 0.471 and 7.48 Hz
(c) 0.666 and 1.35 Hz
(d) 0.666 and 8.50 Hz

(GATE 2009)

Solution. The critical damping constant (Ns/m)

$$c_c = 2\sqrt{km} = 848.528 \text{ Ns/m}$$

Critical damping factor

$$\xi = \frac{c}{c_c} = 0.471405$$

Natural frequency (rad/sec)

$$\omega_n = \sqrt{\frac{k}{m}} = 8.48528 \text{ rad/s}$$

Damped natural frequency

$$\omega_d = \frac{\sqrt{1-\xi^2}\omega_n}{2\pi} = 1.19101 \text{ Hz}$$

Ans. (a)

22. The natural frequency of a spring–mass system on earth is ω_n. The natural frequency of this system on the moon ($g_{moon} = g_{earth}/6$) is

(a) ω_n (b) $0.408\omega_n$
(c) $0.204\omega_n$ (d) $0.167\omega_n$

(GATE 2010)

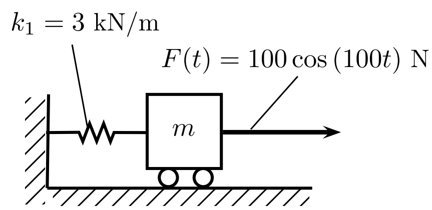
Solution. Natural frequency of spring mass system with stiffness k and mass m is

$$\omega_n = \sqrt{\frac{k}{m}}$$

which is independent of g.

Ans. (a)

23. A mass m attached to a spring is subjected to a harmonic force as shown in the figure.



The amplitude of the forced motion is observed to be 50 mm. The value of m (in kg) is

(a) 0.1 (b) 1.0
(c) 0.3 (d) 0.5

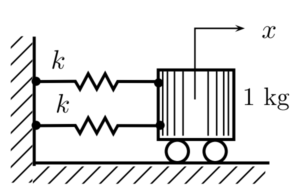
(GATE 2010)

Solution. For equilibrium,

$$F_0 = x_0 k - m\omega^2 x_0$$
$$m = \frac{1}{\omega^2}\left(k - \frac{F_0}{x_0}\right) = 0.1 \text{ kg}$$

Ans. (a)

24. A mass of 1 kg is attached to two identical springs each with stiffness $k = 20$ kN/m as shown in the figure.



Under frictionless condition, the natural frequency of the system in Hz is close to

(a) 32 (b) 23
(c) 16 (d) 11

(GATE 2011)

Solution. Given that

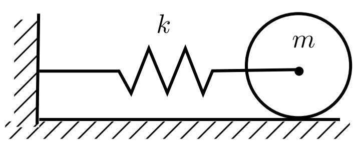
$$k = 20 \times 10^3 \text{ N/m}$$
$$m = 1 \text{ kg}$$

Equivalent stiffness of springs is $2k$, therefore, the natural frequency of vibrations is

$$f = \frac{1}{2\pi}\sqrt{\frac{2k}{m}} = 31.831 \text{ Hz}$$

Ans. (a)

25. A disc of mass m is attached to a spring of stiffness k as shown in the figure.



The disc rolls without slipping on a horizontal surface. The natural frequency (in rad/s) of vibration of the system is

(a) $\sqrt{k/m}$ (b) $\sqrt{2k/m}$
(c) $\sqrt{2k/(3m)}$ (d) $\sqrt{3k/(2m)}$

(GATE 2011)

Solution. Moment of inertia of the disc w.r.t. the base point at circumference is

$$I = \frac{1}{2}mr^2 + mr^2 = \frac{3}{2}mr^2$$

Equilibrium equation is

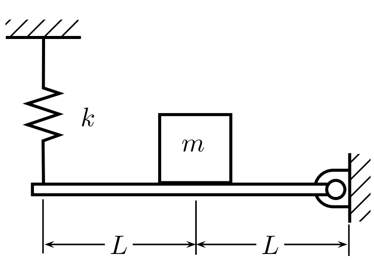
$$\frac{3}{2}mr^2\ddot{\theta} + kr^2\theta = 0$$

Therefore, the natural frequency is

$$\omega_n = \sqrt{\frac{2k}{3m}} \text{ rad/s}$$

Ans. (c)

26. A concentrated mass m is attached at the center of a rod of length $2L$ as shown in the figure.



The rod is kept in a horizontal equilibrium position by a spring of stiffness k. For a very small amplitude of vibration, neglecting the weights of the rod and spring, the undamped natural frequency of the system is

(a) $\sqrt{k/m}$ (b) $\sqrt{2k/m}$
(c) $\sqrt{k/(2m)}$ (d) $\sqrt{4k/m}$

(GATE 2012)

Solution. Equilibrium equation for displacement x of the mass is

$$m\ddot{x} \times L + 2L \times 2x \times k = 0$$
$$m\ddot{x} + 4kx = 0$$

Therefore, the undamped natural frequency of the system is

$$\omega = \sqrt{\frac{4k}{m}}$$

Ans. (d)

27. If two nodes are observed at a frequency of 1800 rpm during whirling of a simply supported long slender rotating shaft, the first critical speed of the shaft in rpm is

(a) 200 (b) 450
(c) 600 (d) 900

(GATE 2013)

Solution. The natural frequency of a simply supported long slender shaft is proportional to the number of modes n. For first critical speed, $n = 1$ for which the frequency is given by

$$f_c = \frac{1}{2} \times 1800 = 900$$

Ans. (d)

28. A single degree of freedom system having mass 1 kg and stiffness 10 kN/mm initially at rest is subjected to an impulse force of magnitude 5 kN for 10^4 s. The amplitude in mm of the resulting free vibration is

(a) 0.5 (b) 1.0
(c) 5.0 (d) 10.0

(GATE 2013)

Solution. The amplitude of vibrations is the initial deflection of the mass, given by

$$x_0 = \frac{F}{k} = 0.5 \text{ mm}$$

Ans. (a)